

Jucys-Murphy operators for Soergel bimodules.

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ABSTRACT. We produce Jucys-Murphy elements for the diagrammatical category of Soergel bimodules associated with general Coxeter groups, and use them to diagonalize the bilinear form on the cell modules. This gives rise to an expression for the determinant of the forms and Jantzen type sum formulas.

1. INTRODUCTION

Soergel bimodules were introduced by Soergel around 1990 in order to prove the Koszul duality conjectures for category \mathcal{O} of a complex simple Lie algebra. This gave rise to a new proof of the Kazhdan-Lusztig conjectures for the composition factor multiplicities of Verma modules. Although Soergel bimodules are essentially combinatorial objects defined in terms of the corresponding Weyl group, the proof of these conjectures relied on intersection homology methods.

Some time later Soergel realised that his bimodule theory makes sense for arbitrary Coxeter systems (W, S) and formulated in this general context the *Soergel conjecture*, relating the indecomposable bimodules with the Kazhdan-Lusztig basis of the associated Hecke algebra. This conjecture was recently proved in a celebrated work by Elias and Williamson. Their proof was entirely algebraic and therefore also provided a proof of the original Kazhdan-Lusztig conjecture avoiding intersection homology. One of the key point of their work was the statement that a certain explicitly defined element ρ acts as a so-called *Lefschetz operator* in the bimodules, with respect to the natural \mathbb{Z} -grading on them.

In positive characteristic and for W infinite, the bimodule category is not well behaved. But over the last decade, starting with the work of Elias and Khovanov, a diagrammatical category \mathcal{D} has been developed that appears to be the correct replacement for the bimodule category in these settings. Apart from the theoretical advantages of \mathcal{D} over the bimodule category, the diagrams themselves also make \mathcal{D} more accessible for calculations than the bimodule category. Indeed Williamson's recent counterexamples to Lusztig's famous conjecture for the representation theory of algebraic groups in characteristic p were first found in \mathcal{D} .

In this paper we study the diagrammatical category \mathcal{D} starting from the fact, proved by Elias and Williamson, that \mathcal{D} is a cellular category. The cellular basis is here a diagrammatical adaption of Libedinsky's light leaves.

The cellular algebra approach to the representation theory of the symmetric group and Hecke algebra in type A was pioneered by Murphy. One of his major insights was the proof that the classical *Jucys-Murphy* elements $\{L_i\}$ act lower triangularly on the cellular basis in these cases. To a large degree it is this triangularity property that explains the importance of the Jucys-Murphy elements in modern representation theory. In fact, it has been axiomatized by Mathas as the defining property of a cellular algebra *endowed with a family of JM-elements*.

The objects of \mathcal{D} are expressions over S , that is words in the alphabet S , and for each such expression \underline{w} we consider the endomorphism algebra $A_{\underline{w}} = \text{End}(\underline{w})$. It is a cellular algebra because \mathcal{D} is a cellular category. The first main result of the paper is our Theorem 6.2 stating that $A_{\underline{w}}$ is a cellular algebra endowed with a family of JM-elements, in the sense of Mathas. Apart from the intrinsic value of this result, we also believe that our JM-elements are of interest by themselves. They are of the form

$$L_i = \left| \begin{array}{c} \text{\scriptsize i'th position} \\ \vdots \\ \cdot \cdot \cdot \begin{array}{c} \color{blue}\uparrow \\ \color{blue}\downarrow \end{array} \cdot \cdot \cdot \\ \vdots \end{array} \right| \quad (1.1)$$

and have actually already appeared in the literature. Indeed, whenever \underline{w} corresponds to a reduced expression of an element of W , we observe that the Lefschetz operator ρ from Elias and Williamson's proof of Soergel's conjecture can be written as a linear combination with coefficients in \mathbb{R}^+ of these L_i 's. Given the importance of \mathbb{R}^+ in Elias and Williamson's work, we find this observation striking.

By general theory, for the cellular algebra $A_{\underline{w}}$ there exists a family of cell modules $\Delta_{\underline{w}}(y)$, each endowed with a bilinear form $\langle \cdot, \cdot \rangle_y$ such that the simple $A_{\underline{w}}$ -modules are of the form $L_{\underline{w}}(y) = \Delta_{\underline{w}}(y) / \text{rad} \langle \cdot, \cdot \rangle_y$. Here the parameter y is an element of W that appears as a subexpression of \underline{w} .

One of the important applications of the triangularity property of the Jucys-Murphy elements, first due to James and Murphy for the symmetric group, is to diagonalize the bilinear form. This gave rise to a cancellation free expression for the determinant of the form. In this paper we show that James and Murphy's idea can be carried out in the setting of \mathcal{D} as well, even though the combinatorics of \mathcal{D} is very different from the Young diagram combinatorics of the symmetric group, of course. Our formula, given in Theorem 9.5 of the paper,

$$\det \langle \cdot, \cdot \rangle_y = \pm \prod_{\beta > 0, s_{\beta} y > y} \beta^{\dim \Delta(s_{\beta} y)} \quad (1.2)$$

has some resemblance with the classical Shapovalov formula for Verma modules.

Following Jantzen's original ideas for Verma modules, we next go on to construct a filtration of $A_{\underline{w}}$ -submodules $\Delta_{\underline{w}}(y) = \Delta_{\underline{w}}^0(y) \supseteq \Delta_{\underline{w}}^1(y) \supseteq \Delta_{\underline{w}}^2(y) \supseteq \dots$ on $\Delta_{\underline{w}}(y)$ and wish to deduce from (1.2) a sum formula for this filtration valid in the Grothendieck group. Here the lack of quasi-heredity of $A_{\underline{w}}$ turns out to be an obstacle for a direct translation of Jantzen's ideas to our setting since the $\Delta_{\underline{w}}(y)$'s do not induce a basis of the corresponding Grothendieck group. We resolve this problem by replacing \underline{w} by a certain subset $\underline{\pi}$ of the expressions over S , containing \underline{w} , and satisfying that the corresponding subset $\pi \subseteq W$ is an ideal in W with respect to the Bruhat order. This gives rise to another cellular algebra $A_{\underline{\pi}}$ with cell modules $\Delta_{\underline{\pi}}(y)$. The algebra $A_{\underline{\pi}}$ is quasi-hereditary as we show in Theorem 9.5, and therefore solves the above mentioned problem.

The above results are valid for any ground field \mathbb{k} and so for any valuation ν on the ground ring R whose fraction field is \mathbb{k} we obtain a filtration $\Delta_{\underline{\pi}}(y) = \Delta_{\underline{\pi}}^0(y) \supseteq \Delta_{\underline{\pi}}^1(y) \supseteq \Delta_{\underline{\pi}}^2(y) \supseteq \dots$ with corresponding sum formula. Although the first term of each of these filtrations $\Delta_{\underline{\pi}}^1(y)$ is always the radical of the form on $\Delta_{\underline{\pi}}(y)$, the other terms of the filtration will in general depend on the valuation ν .

A particularly interesting case of our theory is the case where W is of type A_{n-1} and $\mathbb{k} = \mathbb{F}_p$ since the decomposition numbers for A_{π} in this case are the decomposition numbers for the algebraic group $Sl_n(\overline{\mathbb{F}_p})$ around the Steinberg weight, by Riche and Williamson's recent work. In this case, our sum formula looks as follows

$$\sum_{i \geq 0} [\Delta_{\pi, \mathbb{F}_p}^i(y)] = \sum_{\beta > 0, s_{\beta} y > y} \nu_p(\beta) [\Delta_{\pi, \mathbb{F}_p}(s_{\beta} y)] \quad (1.3)$$

where $\nu_p(\beta)$ is the canonical p -adic valuation, see Theorem 9.8. We give in the end of the paper an example to illustrate how to use this formula to obtain decomposition numbers.

There are several Jantzen type filtrations with associated sum formulas available in the literature, for example Andersen's filtrations for tilting modules. At present, we do not know if it makes sense to ask for possible relations between these filtrations and ours.

The paper is organized as follows. In section 2 we describe the basics of Soergel bimodules. In the sections 3 and 4 we explain the diagrammatical category \mathcal{D} and the diagrammatical category \mathcal{D}^{std} , with an emphasis on the localization methods. Then, in section 5, we describe the cellular basis for \mathcal{D} , the light leaves basis. In section 6 we introduce the L_i 's and verify that they are JM-elements. This relies heavily on the previous sections. We moreover show that they verify a separability condition over the fraction field. In the following section we obtain via the JM-elements a first version of the determinant formula and then, in section 8, we get via Plaza's branching rule the Shapovalov type version of the determinant formula, mentioned above. In section 9 we construct the quasi-hereditary algebra A_{π} and use it to deduce the sum formula from the determinant expression. Finally, in section 10 we give an application of our formula.

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2. THE CATEGORY OF SOERGEL BIMODULES

In this section, we briefly explain the basics of the category of Soergel bimodules.

Let (W, S) be a Coxeter system. Thus W is the group generated by the finite set S subject to the relations $(st)^{m_{st}} = 1$ where $m_{st} = m_{ts}$, $m_{ss} = 1$ and where for $s \neq t$ we have $m_{st} \in \{2, 3, \dots, \infty\}$ ($m_{st} = \infty$ means that there is no relation between s and t).

Let \mathbb{k} be a field. The foundation of the theory of Soergel bimodules is a representation \mathfrak{h} of W over \mathbb{k} which is a *reflection vector faithful* in the sense of Soergel, [So]. By a construction due to Soergel, [So], such a \mathfrak{h} exists for all (W, S) when $\mathbb{k} = \mathbb{R}$ or \mathbb{C} . On the other hand, for infinite W and for \mathbb{k} of positive characteristic it may not exist.

Let us now suppose that \mathfrak{h} is reflection vector faithful and let $R := \oplus_m S^m(\mathfrak{h}^*)$ be the symmetric algebra on \mathfrak{h}^* , which we consider as a \mathbb{Z} -graded algebra with the elements of \mathfrak{h}^* in degree 2. The W -action on \mathfrak{h}^* extends to a W -action on R and for $s \in S$ we denote by R^s the subalgebra $R^s := \{x \in R \mid sx = x\}$. Since the W -action is degree preserving we have that R^s is a \mathbb{Z} -graded subalgebra of R . In the paper 'grading' always means ' \mathbb{Z} -grading' and for any graded module $M = \oplus_i M^i$ we let (1) denote the graded shift defined by $M(1)^i := M^{i+1}$.

Throughout, we shall make a careful distinction between elements w of W and expressions \underline{w} over S representing them. Let \mathbf{exp}_s be the set of expressions over S . In other words \mathbf{exp}_s is the set of words in S ; we denote by \mathbf{rexp}_s the subset of \mathbf{exp}_s consisting of *reduced* expressions in S . There is a canonical map $\mathbf{exp}_s \rightarrow W$, $\underline{w} \mapsto w$ which we often use tacitly. For example, if $s \in S$ and $\underline{w} := ss$ then $\underline{w} \mapsto w = 1$.

For $s \in S$ we define the graded $R \times R$ -bimodule $B_s := R \otimes_{R^s} R(1)$. Let $\underline{w} = s_{i_1} s_{i_2} \dots s_{i_k} \in \mathbf{exp}_s$. Then we define $B_{\underline{w}}$ as

$$B_{\underline{w}} := B_{s_{i_1}} \otimes_R B_{s_{i_2}} \otimes_R \dots \otimes_R B_{s_{i_k}}.$$

This is the Bott-Samelson bimodule associated with \underline{w} . The category of Bott-Samelson bimodules $\mathbb{B}\mathbf{SBim}$ is defined as the category whose objects are sums of shifts of Bott-Samelson bimodules and whose morphisms are homomorphisms of $R \times R$ -bimodules and the category of Soergel bimodules \mathbf{SBim} is the category whose objects are sums of shifts of summands of Bott-Samelson bimodules and whose morphisms are homomorphisms of $R \times R$ -bimodules. In other words, $\mathbf{SBim} := \mathbf{Kar}(\mathbb{B}\mathbf{SBim})$ where \mathbf{Kar} denotes Karoubian envelope. Note that \mathbf{SBim} is not an Abelian category.

Let $\mathcal{H}(W, S)$ be the Hecke algebra associated with (W, S) . Adapting Soergel's conventions, it is the $\mathbb{Z}[q, q^{-1}]$ -algebra with generators $\{H_s | s \in S\}$ subject to the relations $(H_s - q)(H_s + q^{-1}) = 0$ and

$$\underbrace{H_s H_t H_s \dots}_{m_{st} \text{ factors}} = \underbrace{H_t H_s H_t \dots}_{m_{st} \text{ factors}}$$

For $w \in W$ in reduced form $w = s_{i_1} s_{i_2} \dots s_{i_k}$ we define $H_w := H_{i_1} H_{i_2} \dots H_{i_k}$ which by the relations does not depend on the chosen reduced form. Then $\{H_w | w \in W\}$ is a $\mathbb{Z}[q, q^{-1}]$ -basis for $\mathcal{H}(W, S)$. Let $\{\underline{H}_w | w \in W\}$ be the Kazhdan-Lusztig basis for $\mathcal{H}(W, S)$ in the normalization introduced in [So], for example $\underline{H}_s = H_s + q$. The entries of the change of basis matrix between the two bases for $\mathcal{H}(W, S)$ are by definition the Kazhdan-Lusztig polynomials.

Let $\langle \mathbf{SBim} \rangle$ be the split Grothendieck group of \mathbf{SBim} and let $[M]$ denote the class in $\langle \mathbf{SBim} \rangle$ of the object $M \in \mathbf{SBim}$. We make $\langle \mathbf{SBim} \rangle$ into a $\mathbb{Z}[q, q^{-1}]$ -algebra by the rules $[M][N] := [M \otimes_R N]$ and $q[M] := M[-1]$. The following Theorem is known as Soergel's categorification Theorem, it can be found in [So].

Theorem 2.1. *a) For each reduced expression $\underline{w} \in \mathbf{rexp}_s$ of $w \in W$ there is a unique indecomposable bimodule B_w in \mathbf{SBim} that occurs in $B_{\underline{w}}$ (with multiplicity one) and does not occur in B_y for any shorter y . The set $\{B_w(m) | w \in W, m \in \mathbb{Z}\}$ classifies all the indecomposable bimodules in \mathbf{SBim} .
b) There is a unique algebra homomorphism $F : \mathcal{H}(W, S) \rightarrow \langle \mathbf{SBim} \rangle$ given by $\underline{H}_s \mapsto [B_s(1)]$. It is an isomorphism of $\mathbb{Z}[q, q^{-1}]$ -algebras.*

Soergel's conjecture from [So] states that if $\mathbb{k} = \mathbb{C}$, then $F(\underline{H}_w) = [B_w]$. It implies positivity properties for the Kazhdan-Lusztig polynomials and it also implies the Kazhdan-Lusztig conjectures for Verma modules over complex semisimple Lie algebras. It was recently shown by Elias and Williamson, see [EW].

3. THE DIAGRAMMATICAL CATEGORY \mathcal{D} .

Let $A_{\underline{w}} := \text{End}_{\mathbb{B}\mathbf{SBim}}(B_{\underline{w}})$. It is shown in [EW1] that $A_{\underline{w}}$ is a cellular algebra in the sense of Graham and Lehrer. This result is the starting point of our paper and we need to explain it in some detail.

The cellularity of $A_{\underline{w}}$ comes from a diagrammatic realization \mathcal{D} of the bimodule category $\mathbb{B}\mathbb{S}\mathbb{B}\text{im}$. It was introduced by Elias and Williamson in [EW1] in the complete generality we are using here, although preliminary versions of \mathcal{D} already appeared in [EK] and [E]. One advantage of \mathcal{D} over the bimodule category $\mathbb{B}\mathbb{S}\mathbb{B}\text{im}$ is that it is defined for any characteristic, even for infinite groups. Moreover, in positive characteristic \mathcal{D} appears to be the correct category for calculating decomposition numbers for algebraic groups.

The foundation of \mathcal{D} is a *realization* \mathfrak{h} of (W, S) . As indicated above, this is a more general concept than the one of a reflection vector faithful representation.

Definition 3.1. *A realization of (W, S) over a commutative ring \mathbb{k} is a free finite rank module \mathfrak{h} over \mathbb{k} together with sets $\{\alpha_s | s \in S\} \subset \mathfrak{h}^*$ and $\{\alpha_s^\vee | s \in S\} \subset \mathfrak{h}$ such that $\langle \alpha_s^\vee, \alpha_s \rangle = 2$ and such that the rule $s(v) := v - \langle v, \alpha_s \rangle \alpha_s^\vee$ defines a representation of W in \mathfrak{h} . Furthermore, a technical 'balancedness' condition should be verified.*

We shall always assume that Demazure surjectivity holds.

Definition 3.2. *A realization \mathfrak{h} of (W, S) over \mathbb{k} is said to be Demazure surjective if for all $s \in S$ the evaluation maps $\alpha_s : \mathfrak{h} \rightarrow \mathbb{k}$ and $\alpha_s^\vee : \mathfrak{h}^* \rightarrow \mathbb{k}$ are surjective.*

For instance, if \mathbb{k} is of characteristic different from 2, Demazure surjectivity always holds.

Given a realization of \mathfrak{h} of (W, S) , the graded commutative ring R is defined as in the bimodule case. Let us now explain the various other ingredients of \mathcal{D} , as introduced in [EW1].

Definition 3.3. *A diagram for (W, S) (or simply a diagram when confusion is not possible) is a finite diagram on a strip $\mathbb{R} \times [0, 1]$. The arcs are decorated with elements of S . The vertices are the points where the arcs end or meet. The intersection points between the arcs and the upper (lower) border of the strip $\mathbb{R} \times \{1\}$ ($\mathbb{R} \times \{0\}$) are called boundary point and are not vertices. They define sequences of elements of S , called the top sequence and bottom sequence of the diagram. The arcs all end in vertices or boundary points. Loops are allowed. The regions defined by the arcs may be decorated by homogeneous elements of R . Everything is considered up to isotopy.*

The diagram is called standard if the only vertices are $2m_{st}$ -valent, with incident lines of alternating decorations s, t, \dots, s, t .

The diagram is called a Soergel diagram if each vertex is either 0-valent, that is an endpoint of an arc, or 3-valent with the three incident arcs of the same color, or $2m_{st}$ -valent, with incident arcs of alternating decorations s, t, \dots, s, t . The degree of a Soergel diagram is the sum of the degrees of all its vertices and polynomials where a 0-valent vertex has degree 1, a 3-valent vertex has degree -1 and a $2m_{st}$ -valent vertex has degree 0.

When we draw diagrams, we identify S with a set of physical colors and indicate the decorations of the lines by using those colors. Below is an example of a Soergel diagram of degree 5 with $S := \{\text{red}, \text{blue}, \text{green}\}$ and $m_{\text{red}, \text{blue}} = 3$, $m_{\text{red}, \text{green}} = 2$ and $m_{\text{blue}, \text{green}} = 2$.



(3.1)

Standard diagrams are drawn using dashed lines. Below is an example, using the same S and m 's as before.

(3.2)

Definition 3.4. Let \mathcal{D} be the \mathbb{k} -linear monoidal category whose objects are elements of \mathbf{exp}_s . The morphisms $\mathrm{Hom}_{\mathcal{D}}(\underline{w}, \underline{z})$ consist of the free \mathbb{k} -module spanned by Soergel diagrams with bottom sequence \underline{w} and top sequence \underline{z} modulo a number of relations, to be explained below. The composition $g \circ f$ of a morphism $f \in \mathrm{Hom}_{\mathcal{D}}(\underline{w}, \underline{z})$ and a morphism $g \in \mathrm{Hom}_{\mathcal{D}}(\underline{z}, \underline{x})$ is given by vertical concatenation. The monoidal structure is given by horizontal concatenation of expressions.

Let us briefly explain the relations; there are quite a few of them. They are divided into the polynomial relations, the one-color, the two-color and the three-color relations.

Throughout, they are tacitly understood to involve a localized part of the diagrams, leaving the rest unchanged. Also, isotopy of the diagrams is tacitly understood. It corresponds to several relations that combined say that the object \underline{s} is a Frobenius object in \mathcal{D} with adjointness morphisms given by the diagrams  and .

Finally, the cyclicity relation in \mathcal{D} is also tacitly understood throughout. Let D be a diagram with upper rightmost boundary point P of color blue, say, then the multiplication on top of D with the diagram

is an operation called bending downwards the point P . Similarly the operation of bending upwards the lower leftmost point Q is obtained using

(3.3)

The composite operation of bending downwards all the top points of D and bending upwards all the bottom points of D , is called rotation by 180° of D and performing it twice is called rotation by 360° of D . The cyclicity relation states that rotation by 360° of D is equal to D itself for all D .

The polynomial and one-color relations are as follows (for the color blue)

(3.4)

$$f \left| \begin{array}{c} \bullet \\ \downarrow \end{array} \right. = \left| \begin{array}{c} \bullet \\ \downarrow \end{array} \right. s f + \left| \begin{array}{c} \bullet \\ \downarrow \end{array} \right. \partial f \quad (3.5)$$

$$\begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} = \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \quad (3.6)$$

$$\begin{array}{c} \text{---} \bullet \\ | \end{array} = \begin{array}{c} | \end{array} \quad (3.7)$$

$$\bigcirc = 0 \quad (3.8)$$

where $\partial : R \rightarrow R$ is the Demazure operator given by

$$\partial(f) := \frac{f - s(f)}{\alpha}.$$

Here horizontal lines are a shorthand notation for arcs that either go upwards or downwards. For instance the relation (3.6) above actually corresponds to two different relations and similarly for the relation (3.6).

We do not explain the remaining relations, but refer the reader to [EW1].

Let us now suppose that \mathfrak{h} is a reflection vector faithful representation of (W, S) such that the bimodule category $\mathbb{B}\mathbb{S}\text{Bim}$ exists and is well behaved, in the sense that Soergel's categorification Theorem 2.1 holds. Then there is a functor $\mathcal{F} : \mathcal{D} \rightarrow \mathbb{B}\mathbb{S}\text{Bim}$, defined on objects by

$$\mathcal{F}(\underline{w}) := B_{\underline{w}}$$

where $B_{\emptyset} := R$. On one-color morphisms \mathcal{F} is defined as follows

$$\left| \begin{array}{c} \bullet \\ \downarrow \end{array} \right. \mapsto f \otimes g \mapsto fg \quad (3.9)$$

$$\left| \begin{array}{c} \bullet \\ \downarrow \end{array} \right. \mapsto 1 \mapsto \frac{1}{2}(\alpha \otimes 1 + 1 \otimes \alpha) \quad (3.10)$$

$$\begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \mapsto 1 \otimes g \otimes 1 \mapsto \partial g \otimes 1 \quad (3.11)$$

$$\begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \mapsto 1 \otimes 1 \mapsto 1 \otimes 1 \otimes 1. \quad (3.12)$$

Here, if \mathbb{k} is of characteristic 2, one should replace $\frac{1}{2}(\alpha \otimes 1 + 1 \otimes \alpha)$ in (3.10) by $\Delta_s := \delta \otimes 1 - 1 \otimes s(\delta)$ where δ is defined by $\partial(\delta) = 1$; it exists because of Demazure surjectivity.



On polynomials, \mathcal{F} is defined by mapping polynomial $f \in R$ to multiplication by f in the slot of $B_{\underline{w}}$ associated with the region of f . We do not here explain the image of \mathcal{F} on diagrams involving more colours but refer once again the reader to [EW1].

If \mathbb{k} a field it follows by Elias and Williamson's work [EW1] that \mathcal{F} is an equivalence of categories since $\mathbb{S}\text{Bim}$ is well behaved. In other words, the quest for indecomposable Soergel bimodules in $\mathbb{S}\text{Bim}$ is equivalent to the quest for primitive idempotents in $A_{\underline{w}}$ for all \underline{w} .

4. THE DIAGRAMMATICAL CATEGORY \mathcal{D}^{std} .

In this section we explain the diagrammatic category \mathcal{D}^{std} . It is an auxiliary category with a particularly simple structure that we shall rely on for our main results. The relationship between \mathcal{D} and \mathcal{D}^{std} is comparable to the relationship between the category of finite dimensional $\mathbb{F}_p\mathfrak{S}_n$ -modules and the category of finite dimensional $\mathbb{Q}\mathfrak{S}_n$ -modules, where \mathfrak{S}_n is the symmetric group on n letters.

Definition 4.1. \mathcal{D}^{std} is the additive \mathbb{k} -linear monoidal category whose objects are direct sums of elements of \mathbf{exp}_s and whose morphisms $\text{Hom}_{\mathcal{D}^{std}}(\underline{w}, \underline{z})$ consist of the free \mathbb{k} -module spanned by standard diagrams with bottom sequence \underline{w} and top sequence \underline{z} modulo a number of relations. The composition of morphisms and the monoidal structure on \mathcal{D}^{std} are defined as in \mathcal{D} .

Let us briefly explain the relations. One should think of them as the relations in \mathcal{D} with the 'lower terms' deleted. The tacit relations from \mathcal{D} carry over to \mathcal{D}^{std} . For example, the diagrams  and  form an adjoint pair such that the cyclicity relation holds with respect to them.

In \mathcal{D}^{std} we have the following one-color relations

$$\left| \begin{array}{c} \vdots \\ f \\ \vdots \end{array} \right| = \left| \begin{array}{c} \vdots \\ sf \\ \vdots \end{array} \right| \quad (4.1)$$

$$\begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array} = 1 \quad (4.2)$$

$$\begin{array}{c} \cup \\ \cap \end{array} = \left| \begin{array}{c} \vdots \\ \vdots \end{array} \right|. \quad (4.3)$$

The two-color relations look in \mathcal{D}^{std} as follows (in the cases $m = 2, m = 3$)

$$\begin{array}{c} \text{X} \\ \text{X} \end{array} = \left| \begin{array}{c} \vdots \\ \vdots \end{array} \right|, \quad \begin{array}{c} \text{X} \\ \text{X} \end{array} = \left| \begin{array}{c} \vdots \\ \vdots \end{array} \right| \quad (4.4)$$

that is, the $2m$ -valent vertices are idempotents.

We skip the explanation of the three-color relations of \mathcal{D}^{std} , once more referring the reader to [EW1]. What is important is the fact that \mathcal{D}^{std} is a much simpler category than \mathcal{D} . Indeed, Elias and Williamson proved in [EW2] that

$$\text{Hom}_{\mathcal{D}^{std}}(\underline{x}, \underline{y}) = \begin{cases} \mathbb{k} & \text{if } \underline{x} = \underline{y} \\ 0 & \text{otherwise.} \end{cases} \quad (4.5)$$

For $\underline{y}, \underline{y}' \in \mathbf{exp}_s$ with $y = y'$ there is a unique diagram $Std(\underline{y}, \underline{y}')$ in \mathcal{D}^{std} from \underline{y} to \underline{y}' . The basis for $\text{Hom}_{\mathcal{D}^{std}}(\underline{y}, \underline{y}')$ consists of that diagram $Std(\underline{y}, \underline{y}')$.

In the case of \mathfrak{h} being a reflection vector faithful representation of (W, S) there is a bimodule category equivalent to \mathcal{D}^{std} . Indeed, for $\underline{w} \in \mathbf{exp}_s$ we define the *standard* R -bimodule R_w as R itself as a ring, where $f \in R$ acts on the left via multiplication by f , and acts on the right via multiplication by $w(f)$. Note that R_w is not a Soergel bimodule. We define $StdBim$ to be the \mathbb{k} -linear, monoidal, additive category whose objects are sums of standard R -bimodules. One can then show that

$$\text{Hom}_{StdBim}(R_{\underline{x}}, R_{\underline{y}}) = \begin{cases} \mathbb{k} & \text{if } x = y \\ 0 & \text{otherwise.} \end{cases}$$

Combining this with (4.5) it follows that \mathcal{D}^{std} and $StdBim$ are equivalent categories via the functor that sends \underline{x} to $R_{\underline{x}}$.

Let $Q := Q(R)$ be the quotient field of R and let \mathcal{D}_Q^{std} be the corresponding category of standard diagrams. There is another diagrammatic way of presenting \mathcal{D}_Q^{std} , which we shall need. Indeed, in the bimodule setting we have, working over Q , that $B_s = Q_s \oplus Q_1$. Let $\iota : Q_s \rightarrow B_s$ and $\pi : B_s \rightarrow Q_s$ denote the splitting maps. We represent them by the following diagrams, containing *bivalent* vertices:

$$\iota = \begin{array}{c} | \\ \vdots \\ \bullet \end{array} \quad \pi = \begin{array}{c} \vdots \\ | \\ \bullet \end{array} \quad (4.6)$$

These diagrams are the starting point for the monoidal category $Kar(\mathcal{D}_Q)$, also introduced in [EW1]. The name is justified by the Theorem to follow. The objects of $Kar(\mathcal{D}_Q)$ are expressions \underline{w} over S just as before, but where each letter $s \in S$ may now be a normal index or a *reflection index*. The morphisms of the normal indices are drawn normally, whereas the morphisms of the reflection indices are drawn using dashed arcs.

The morphisms in $\text{Hom}_{Kar(\mathcal{D}_Q)}(\underline{w}, \underline{y})$ are spanned over Q by Soergel diagrams from \underline{w} to \underline{y} allowing normal arcs and dashed arcs. Bivalent vertices, as illustrated above, are allowed.

The relations in $Kar(\mathcal{D}_Q)$ involving undashed morphisms are the same as in \mathcal{D}_Q and the relations involving dashed morphisms are as in \mathcal{D}_Q^{std} . The relations in $Kar(\mathcal{D}_Q)$ involving bivalent vertices are as follows (for the color blue)

$$\begin{array}{c} \bullet \\ | \\ \vdots \\ \bullet \end{array} = 0 \quad \begin{array}{c} \vdots \\ | \\ \bullet \end{array} = 0 \quad (4.7)$$

$$\begin{array}{c} | \\ \vdots \\ \bullet \end{array} = \frac{1}{\alpha} \begin{array}{c} \bullet \\ | \\ \bullet \end{array} + \frac{1}{\alpha} \begin{array}{c} | \\ \vdots \\ \bullet \end{array} \quad (4.8)$$

$$\begin{array}{c} \vdots \\ | \\ \vdots \end{array} = \alpha \begin{array}{c} \vdots \\ | \\ \vdots \end{array} \quad (4.9)$$

$$\begin{array}{c} | \\ \vdots \\ \bullet \end{array} = \alpha \begin{array}{c} | \\ \vdots \\ \bullet \end{array} - \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \quad (4.10)$$

$$\text{dashed arc} = \frac{1}{\alpha} \text{solid arc} \quad (4.11)$$

$$\text{dashed cup} = \frac{1}{\alpha} \text{solid cup} \quad (4.12)$$

One now gets, using the above relations, that rotation by 360 degrees maps a bivalent vertex to its negative and hence $Kar(\mathcal{D}_Q)$ is *not* a cyclic category (but almost).

For the m -valent vertex we impose the following relation (in the case $m = 3$)

$$\text{dashed vertex} = \frac{1}{\rho} \text{solid vertex} \quad (4.13)$$

where ρ is the product of the positive roots of the corresponding dihedral group. Actually, this can be viewed as the definition of the dashed m -valent vertex. By the balancedness condition mentioned above it is stable under color-preserving rotations.

There is a natural functor $\mathcal{F} : \mathcal{D}_Q^{std} \rightarrow Kar(\mathcal{D}_Q)$ that on objects maps direct sums of expressions to themselves and on morphism maps (dashed) diagrams to dashed diagrams. Elias and Williamson showed in [EW1] that \mathcal{F} is an equivalence of categories:

Theorem 4.2. *\mathcal{F} is an equivalence of categories. Moreover $Kar(\mathcal{D}_Q)$ is equivalent to the Karoubian envelope of \mathcal{D}_Q^{std} (the objects \underline{w} of \mathcal{D}_Q^{std} are indecomposable) and to the bimodule category $StdBim$.*

The inverse functor \mathcal{G} applied to a diagram D is obtained using relation (4.8) on all undashed arcs of D and then reducing D to a linear combination of standard diagrams, via Theorem 4.5.

5. CELLULARITY OF \mathcal{D} .

Let us now return to the category \mathcal{D} . Elias and Williamson showed in Proposition 6.23 of [EW] that it is a cellular category, in particular $A_{\underline{w}} = \text{End}_{\mathcal{D}}(B_{\underline{w}})$ is a cellular algebra for all \underline{w} . In this section we explain the various ingredients of the cellular structure of \mathcal{D} according to [EW]. The cellular basis itself is a diagrammatic version of Libedinsky's light leaves basis [L].

Let us first recall the definition of a cellular algebra first formulated by Graham and Lehrer in [GL].

Definition 5.1. *Let A be a finite dimensional algebra over a commutative ring k . Then a cellular basis for A is a triple (Λ, Tab, C) where Λ is a poset, Tab is a function from Λ to sets and $C : \coprod_{\lambda \in \Lambda} \text{Tab}(\lambda) \times \text{Tab}(\lambda) \rightarrow A$ is an injection such that*

$$\{C_{st}^\lambda \mid s, t \in \text{Tab}(\lambda), \lambda \in \Lambda\}$$

is a k -basis for A , denoted the cellular basis. The rule $(C_{st}^\lambda)^ := C_{ts}^\lambda$ defines a k -linear antihomomorphism of A and, finally, the structure constants for A with*

respect to $\{C_{st}^\lambda\}$ satisfy the following condition with respect to the partial order: for all $a \in A$ we have

$$aC_{st}^\lambda = \sum_{u,t \in \text{Tab}(\lambda)} r_{usa} C_{ut}^\lambda + \text{lower terms}$$

where lower terms means a linear combination of C_{ab}^μ where $\mu < \lambda$ and where $r_{usa} \in k$.

Suppose that $\underline{w} = s_{i_1} s_{i_2} \dots s_{i_k} \in \mathbf{exp}_s$. We then define a *subexpression* of \underline{w} to be a sequence $\mathbf{e} = (e_1, e_2, \dots, e_k) \in \{0, 1\}^k$. For each such subexpression \mathbf{e} we define $\underline{w}^{\mathbf{e}} := s_{i_1}^{e_1} s_{i_2}^{e_2} \dots s_{i_k}^{e_k}$ and we define $w^{\mathbf{e}} \in W$ the corresponding group element. If $w^{\mathbf{e}} = y$ then \mathbf{e} is said to express y .

Let $\underline{w}^{\mathbf{e}} = s_{i_1}^{e_1} s_{i_2}^{e_2} \dots s_{i_k}^{e_k}$. Let us now explain the various ingredients of the cellular basis for $\underline{A}_{\underline{w}}$ (and \mathcal{D}). For k we choose R itself. For Λ we choose $\{w^{\mathbf{e}} | \mathbf{e} \text{ subexpression of } \underline{w}\}$ with the poset structure induced by the usual Bruhat order on W (having 1 as the smallest element).

For the Tab function we choose

$$\text{Tab}(y) := \{\mathbf{e} \in \{0, 1\}^k | \mathbf{e} \text{ subexpression of } \underline{w} \text{ expressing } y\}$$

and for the antihomomorphism we choose the reflection along a horizontal axis.

We are now only left with the definition of the cellular basis itself. We need some combinatorial concepts related to subexpressions. For $\underline{w} = s_{i_1} \dots s_{i_j} \dots s_{i_k}$ we define $\underline{w}^{\leq j} = s_{i_1} \dots s_{i_j} \in \mathbf{exp}_s$. Any subexpression \mathbf{e} of \underline{w} defines a sequence $(\underline{w}_0, \underline{w}_1, \dots, \underline{w}_k)$ in \mathbf{exp}_s via $\underline{w}_0 = 1$ and recursively

$$\underline{w}_j = \begin{cases} \underline{w}_{j-1} s_j & \text{if } e_j = 1 \\ \underline{w}_{j-1} & \text{otherwise.} \end{cases}$$

This gives rise to a series of *symbols* $T = (t_1, t_2, \dots, t_k) \in \{U, D\}^k$ ($U = \text{up}$, $D = \text{down}$) defined as follows

$$t_j = \begin{cases} U & \text{if } w_{j-1} s_j > w_{j-1} \\ D & \text{otherwise.} \end{cases}$$

In particular, we always have that $t_1 = U$. We merge the sequences T and \mathbf{e} into one sequence $M = (m_1, m_2, \dots, m_k)$ by concatenating the symbols, that is $M_j := t_j e_j$.

With this notation at hand, we construct a series of morphism $\mathbb{L}\mathbb{L}_{\underline{w}, \mathbf{e}, \leq j} \in \text{Hom}_{\mathcal{D}}(B_{\underline{w}^{\leq j}}, B_{\underline{w}_j})$ as follows. We set first $\mathbb{L}\mathbb{L}_{\underline{w}, \mathbf{e}, \leq 0}$ to be the empty diagram. Suppose now that $\alpha := \mathbb{L}\mathbb{L}_{\underline{w}, \mathbf{e}, \leq j-1}$ has already been constructed. Then $\mathbb{L}\mathbb{L}_{\underline{w}, \mathbf{e}, \leq j}$ is obtained from α by first adding on the right a vertical arc of color i_j . This arc is then further manipulated in a way that depends on the value of M_j . The rules are as follows.

If $M_j = U0$, then the new arc is terminated with a dot.

If $M_j = U1$, then the new arc is continued to the top.

If $M_j = D0$, then s_{i_j} is in the right descent set of w_{k-1} and one applies a series of m -valent vertices to α such that the result has an arc of color i_j to the right of α . Finally a trivalent vertex of color i_j is applied to the final two strands.

If $M_j = D1$, then we proceed as in case $D0$ but finish with a cap of color i_j .

Let us illustrate the four cases:

$$\begin{array}{cc}
 U0 : \begin{array}{c} \text{---} \\ \text{---} \\ \boxed{\alpha} \\ \text{---} \\ \text{---} \end{array} \quad U1 : \begin{array}{c} \text{---} \\ \text{---} \\ \boxed{\alpha} \\ \text{---} \\ \text{---} \end{array} \\
 \text{---} \quad \text{---} \quad \text{---} \quad \text{---}
 \end{array} \tag{5.1}$$

$$\begin{array}{cc}
 D0 : \begin{array}{c} \text{---} \\ \text{---} \\ \boxed{\alpha} \\ \text{---} \\ \text{---} \end{array} \quad D1 : \begin{array}{c} \text{---} \\ \text{---} \\ \boxed{\alpha} \\ \text{---} \\ \text{---} \end{array} \\
 \text{---} \quad \text{---}
 \end{array} \tag{5.2}$$

Finally one defines $\mathbb{LL}_{\underline{w}, \mathbf{e}} := \mathbb{LL}_{\underline{w}, \mathbf{e}, \leq k}$: this is the *light leaves morphism* associated with \mathbf{e} . It is a diagrammatical version of the bimodule homomorphism introduced by Libedinsky, see [Li].

For all $\mathbf{e} \in \text{Tab}(y)$ our chosen $\mathbb{LL}_{\underline{w}, \mathbf{e}}$ belongs to $\text{Hom}_{\mathcal{D}}(B_{\underline{w}}, B_{\underline{w}^{\mathbf{e}}})$ where $\underline{w}^{\mathbf{e}} = y$. We choose a fixed expression \underline{y} for y and fix for all appearing $\underline{w}^{\mathbf{e}}$ a set of braid moves transforming $\underline{w}^{\mathbf{e}}$ to \underline{y} . We then modify the $\mathbb{LL}_{\underline{w}, \mathbf{e}}$ by multiplying with the corresponding set of m -valent vertices. In this way, all chosen $\mathbb{LL}_{\underline{w}, \mathbf{e}}$ now belong to the same $\text{Hom}_{\mathcal{D}}(B_{\underline{w}}, B_{\underline{y}})$.

It should be noted that although the $\mathbb{LL}_{\underline{w}, \mathbf{e}}$'s depend heavily on the choices of m -valent vertices along the way, any choices will do.

For $\mathbf{e}, \mathbf{e}_1 \in \text{Tab}(y)$ we define $\beta := \mathbb{LL}_{\underline{w}, \mathbf{e}}$ and $\beta_1 := \mathbb{LL}_{\underline{w}, \mathbf{e}_1}$. Let $\beta_1^* \in \text{Hom}_{\mathcal{D}}(B_{\underline{y}}, B_{\underline{w}})$ be the morphism obtained from β_1 by reflection along a horizontal axis. We then define the last ingredient C of the cellular basis as $C_{\mathbf{e}\mathbf{e}_1}^y = \mathbb{LL}_{\mathbf{e}, \mathbf{e}_1, y} := \beta_1^* \beta \in A_{\underline{w}}$.

Let us illustrate the light leaves basis in a couple of very simple examples. Let first W be of type A_1 , that is $S = \{s\}$. Then W has just two elements and all Soergel diagrams are one-colored, say blue. Let us take $\underline{w} := ss$ representing 1. Then we have

$$\begin{aligned}
 \text{Tab}(1) &= \{(0, 0), (1, 1)\} \\
 \text{Tab}(s) &= \{(0, 1), (1, 0)\}
 \end{aligned}$$

with corresponding symbols

$$\begin{aligned}
 &\{(U0, U0), (U1, D1)\} \\
 &\{(U0, U1), (U1, D0)\}.
 \end{aligned}$$

This gives rise to the following diagrams in $\text{Tab}(1)$

$$\begin{array}{c} \text{---} \end{array} \quad \begin{array}{c} \text{---} \end{array} \tag{5.3}$$

whereas in $\text{Tab}(s)$ we get

$$\begin{array}{c} \text{---} \end{array} \quad \begin{array}{c} \text{---} \end{array} \tag{5.4}$$

Thus $A_{\underline{w}}$ is of dimension 8 with cellular basis given by the diagrams

$$\begin{array}{cccc}
 \begin{array}{c} \downarrow \downarrow \\ \downarrow \downarrow \end{array} & \begin{array}{c} \downarrow \downarrow \\ \cap \end{array} & \begin{array}{c} \cup \\ \downarrow \downarrow \end{array} & \begin{array}{c} \cup \\ \cap \end{array} \\
 \end{array} \quad (5.5)$$

and

$$\begin{array}{cccc}
 \begin{array}{c} \downarrow \downarrow \\ \downarrow \downarrow \end{array} & \begin{array}{c} \downarrow \downarrow \\ \downarrow \downarrow \end{array} & \begin{array}{c} \downarrow \downarrow \\ \downarrow \downarrow \end{array} & \begin{array}{c} \downarrow \downarrow \\ \downarrow \downarrow \end{array} \\
 \end{array} \quad (5.6)$$

Let us next take $\underline{w} := s s s$ representing s . Then we have

$$\begin{aligned}
 \text{Tab}(1) &= \{(0, 0, 0), (1, 1, 0), (0, 1, 1), (1, 0, 1)\} \\
 \text{Tab}(s) &= \{(0, 0, 1), (1, 1, 1), (0, 1, 0), (1, 0, 0)\}.
 \end{aligned}$$

The corresponding symbols M are

$$\begin{aligned}
 &\{(U0, U0, U0), (U1, D1, U0), (U0, U1, D1), (U1, D0, D1)\} \\
 &\{(U0, U0, U1), (U1, D1, U1), (U0, U1, D0), (U1, D0, D0)\}
 \end{aligned} \quad (5.7)$$

and so the corresponding light leaves diagrams in $\text{Tab}(1)$ are

$$\begin{array}{cccc}
 \begin{array}{c} \downarrow \downarrow \downarrow \\ \downarrow \downarrow \downarrow \end{array} & \begin{array}{c} \cap \downarrow \downarrow \end{array} & \begin{array}{c} \downarrow \downarrow \cap \end{array} & \begin{array}{c} \cap \cap \end{array} \\
 \end{array} \quad (5.8)$$

and in $\text{Tab}(s)$ they are

$$\begin{array}{cccc}
 \begin{array}{c} \downarrow \downarrow \downarrow \\ \downarrow \downarrow \downarrow \end{array} & \begin{array}{c} \cap \downarrow \downarrow \end{array} & \begin{array}{c} \downarrow \downarrow \cap \end{array} & \begin{array}{c} \cap \cap \end{array} \\
 \end{array} \quad (5.9)$$

Thus the cellular basis for $\text{End}_{\mathcal{D}}(B_{\underline{w}})$ has 32 elements. Here are four of them corresponding to $\text{Tab}(1) \times \text{Tab}(1)$

$$\begin{array}{cccc}
 \begin{array}{c} \downarrow \downarrow \downarrow \\ \downarrow \downarrow \downarrow \end{array} & \begin{array}{c} \downarrow \downarrow \downarrow \\ \downarrow \downarrow \downarrow \end{array} & \begin{array}{c} \downarrow \downarrow \downarrow \\ \downarrow \downarrow \downarrow \end{array} & \begin{array}{c} \downarrow \downarrow \downarrow \\ \downarrow \downarrow \downarrow \end{array} \\
 \end{array} \quad (5.10)$$

Let us finish this section with an example of type A_2 , that is $S := \{\alpha, \beta\}$ with $m_{\alpha, \beta} = 3$. Let us take $s_1 := \alpha$ and $s_2 := \beta$ and let $\underline{w} := s_1 s_2 s_1$. Then $\text{Tab}(1) = \{(0, 0, 0), (1, 0, 1)\}$ with symbols $\{(U0, U0, U0), (U1, U0, D1)\}$ and the corresponding light leaves morphisms are

$$\begin{array}{cc}
 \begin{array}{c} \downarrow \downarrow \downarrow \\ \downarrow \downarrow \downarrow \end{array} & \begin{array}{c} \cap \downarrow \downarrow \end{array} \\
 \end{array} \quad (5.11)$$

We have $\text{Tab}(s_1) = \{(1, 0, 0), (0, 0, 1)\}$ with symbols $\{(U1, U0, D0), (U0, U0, U1)\}$ and light leaves morphisms

$$\begin{array}{cc}
 \begin{array}{c} \cap \downarrow \downarrow \end{array} & \begin{array}{c} \downarrow \downarrow \cap \end{array} \\
 \end{array} \quad (5.12)$$

We have $\text{Tab}(s_1 s_2) = \{(1, 1, 0)\}$ with symbols $\{(U1, U1, U0)\}$ and light leaf

$$\begin{array}{c} \downarrow \downarrow \downarrow \\ \downarrow \downarrow \downarrow \end{array} \quad (5.13)$$

Let us finally mention $\text{Tab}(s_1 s_2 s_1) = \{(1, 1, 1)\}$ that corresponds to the identity map

$$\begin{array}{c} | \\ | \\ | \end{array} \quad \begin{array}{c} | \\ | \\ | \end{array} \quad \begin{array}{c} | \\ | \\ | \end{array} \quad . \quad (5.14)$$

6. JUCYS-MURPHY ELEMENTS

Let us fix an expression $\underline{w} = s_{i_1} s_{i_2} \dots s_{i_k} \in \mathbf{exp}_s$. In this section we consider for $i \in \{1, 2 \dots k\}$ the diagram $L_j \in A_{\underline{w}}$ given by

$$L_j = \begin{array}{c} \text{\textit{j}'th position} \\ \left| \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right| \cdots \left| \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right| \end{array} \quad (6.1)$$

where the first color black refers to i_1 , the second color black refers to i_2 and so on, except that the blue color refers to i_j . In this section we study the left multiplication by L_j on the light leaves basis $\{\mathbb{L}_{\mathbf{e}, \mathbf{e}_1, y}\}$. It is interesting to note that the elements L_j already appear, somewhat hidden, in Elias and Williamson's proof of Soergel's conjecture, see [EW]. Indeed, suppose that $\underline{w} = s_{i_1} s_{i_2} \dots s_{i_k}$ and set $\overline{B_{\underline{w}}} := B_{\underline{w}} \otimes_R \mathbb{R}$ where \mathbb{R} is made into an R algebra by mapping the positive degree elements to zero. Then the Lefschetz element ρ introduced in [EW], acting in $\overline{B_{\underline{w}}}$, satisfies by Lemma 3.4 of [EW] the formula

$$\rho(-) = \sum_{j=1}^k (s_{i_{j-1}} \dots s_{i_1} \rho)(\alpha_{i_j}^{\vee}) \iota_j \circ \phi_j. \quad (6.2)$$

But $\begin{array}{c} | \\ | \\ | \end{array}$ is the diagrammatical version of ϕ_j and $\begin{array}{c} | \\ | \\ | \end{array}$ is the diagrammatical version of ι_j . Thus, if \underline{w} is reduced expression, then we have that ρ as an operator of the Bott-Samelson bimodule $\overline{B_{\underline{w}}}$ is an \mathbb{R}^+ -linear combination of our L_i 's.

Our main result of this section is that the set $\{L_j | j = 1, \dots, k\}$ verifies the abstract condition, due to Mathas in [Ma], for being a set of Jucys-Murphy elements for $A_{\underline{w}}$ with respect to the light leaves basis.

Let us recall the definition of a set of JM-elements, [Ma]. Note that the classical theory of Jucys-Murphy elements was developed mostly by Murphy in the eighties, see [M81], [M83], [M92], [M95], and that the definition of JM-element can be considered as an axiomatization of this theory.

Definition 6.1. *Let A be a cellular algebra over k with datum (Λ, Tab, C) . For each $\lambda \in \Lambda$, suppose that $\text{Tab}(\lambda)$ is endowed with a poset structure with order relation $<$ (depending on λ). Let $\mathbf{L} = \{L_1, L_2, \dots, L_k\}$ be a commutative family of elements of A , such that $L_i^* = L_i$ for all i . Suppose that there is a function $c_a : \{1, 2, \dots, k\} \rightarrow k$ for each $a \in \text{Tab}(\lambda)$. Then \mathbf{L} is said to be a family of JM-elements for A with content functions $\{c_a\}$ if for all i we have that*

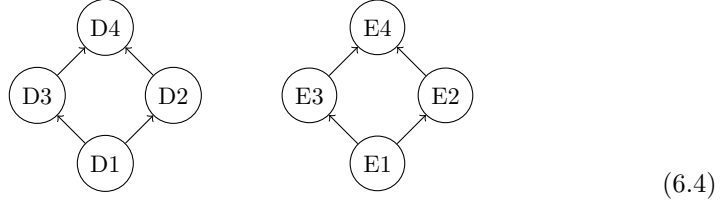
$$L_i C_{ab}^{\lambda} = c_a(i) C_{ab}^{\lambda} + \text{lower terms} \quad (6.3)$$

where lower terms means a linear combination of elements from $\{C_{a_1 b}^{\lambda} | a_1 < a\} \cup \{C_{a_2 b_2}^{\mu} | \mu < \lambda\}$.

For the partial order on $\text{Tab}(y)$ of our cellular basis for $A_{\underline{w}}$ we use the path dominance order \preceq introduced by Elias and Williamson. Let us explain it. Recall that a subexpression \mathbf{e} of \underline{w} defines a sequence $(\underline{w}_1, \underline{w}_2, \dots, \underline{w}_k)$ in \mathbf{exp}_s and hence also a sequence (w_1, w_2, \dots, w_k) of elements in W . If \mathbf{f} is another subexpression of \underline{w} with corresponding sequence (v_1, v_2, \dots, v_k) in W then we say that $\mathbf{e} \preceq \mathbf{f}$ if $w_i \leq v_i$ for all i . If $\mathbf{e} \preceq \mathbf{f}$ we also say $L_{\overline{w}, \mathbf{e}} \preceq L_{\overline{w}, \mathbf{f}}$.

In the $A1$ examples (5.3) and (5.4) above we have that the first diagram is less than the second, with respect to the path dominance order \preceq .

Let us consider the diagrams of (5.8). Let us denote them $D1, D2, D3, D4$ from left to right and let similarly the diagrams of (5.9) be denoted $E1, E2, E3, E4$. Then these diagrams are related via path dominance \preceq as follows



with $D1$ and $E1$ being the smallest and so on.

Recall that $\underline{w} = s_{i_1} s_{i_2} \dots s_{i_k}$ is our fixed expression for w . Suppose that $\mathbf{e} = (e_1, e_2, \dots, e_k) \in \text{Tab}(y)$ and that T is the a 'th coordinate of the string of symbols given by \mathbf{e} , that is $T \in \{U0, U1, D0, D1\}$. We then define for $a = 1, 2, \dots, k$ our candidate for the content function $c_{\mathbf{e}} : \{1, 2, \dots, k\} \rightarrow k$ as follows

$$c_{\mathbf{e}}(a) := \begin{cases} 0 & \text{if } T = U1, D1 \\ s_{i_1}^{e_1} \dots s_{i_{a-1}}^{e_{a-1}} \alpha_{i_a} & \text{if } T = U0, D0. \end{cases} \quad (6.5)$$

We are now in position to state and prove the promised Theorem. The proof is based on Proposition 6.6 of [EW] which is used there to show the linear independence of the light leaves basis.

Theorem 6.2. *The set $\mathbf{L} := \{L_1, \dots, L_k\}$ defined in (6.1) is a set of JM-elements for $A_{\underline{w}}$ with respect to the light leaves basis and path dominance order \preceq .*

Proof: Clearly the L_i 's commute and satisfy $L_i^* = L_i$ and hence we only need to verify the lower triangularity condition (6.3). Recall that we have fixed a target object \underline{y} for all light leaves morphism in $\{\mathbb{L}\mathbb{L}_{\underline{w}, \mathbf{e}} | \mathbf{e} \in \text{Tab}(y)\}$. Recall also the equivalence of categories $\mathcal{F} : \mathcal{D}_Q^{std} \rightarrow \text{Kar}(\mathcal{D}_Q)$ with inverse functor G . For any $L_{\underline{w}, \mathbf{e}}$ with $\mathbf{e} \in \text{Tab}(y)$ we have that $G(L_{\underline{w}, \mathbf{e}})$ is a Q -linear combination of standard morphisms $Std(\underline{z}, \underline{z}')$ such that $z = z'$ and such that, by the recursive construction of the light leaves, if $\underline{z} = \underline{w}^{\mathbf{f}}$ then $\mathbf{f} \preceq \mathbf{e}$ and if $\underline{z}' = \underline{y}^{\mathbf{f}_1}$ then \mathbf{f}_1 is a subexpression of \underline{y} .

In the proof of Proposition 6.6 in [EW] it is proved that if $\underline{z} = \underline{w}^{\mathbf{e}}$ and if $\underline{z}' = \underline{y}$ then the coefficient of $Std(\underline{z}, \underline{z}')$ is a unit in Q . Hence the Q -spans of the following two sets coincide

$$\{G(\mathbb{L}\mathbb{L}_{\underline{w}, \mathbf{f}}) | \mathbf{f} \preceq \mathbf{e}\} \text{ and } \{Std(\underline{w}^{\mathbf{f}}, \underline{y}^{\mathbf{f}_1}) | \mathbf{f} \preceq \mathbf{e}, \mathbf{f}_1 \text{ subexpression of } \underline{y}\}. \quad (6.6)$$

Expanding, we then get

$$G(\mathbb{L}\mathbb{L}_{\underline{w}, \mathbf{e}}) = \sum_{\substack{\mathbf{f}_1 \\ \mathbf{f} \preceq \mathbf{e}}} a_{\mathbf{f}, \mathbf{f}_1} Std(\underline{w}^{\mathbf{f}}, \underline{y}^{\mathbf{f}_1}) \quad (6.7)$$


where $a_{\mathbf{f}, \mathbf{f}_1} \in Q$ and \mathbf{f}_1 runs over subexpressions of \underline{y} . From this we get that


$$L_i G(\mathbb{L}\underline{w}, \mathbf{e}) = G(L_i \mathbb{L}\underline{w}, \mathbf{e}) = \sum_{\substack{\mathbf{f}_1 \\ \mathbf{f} \preceq \mathbf{e}}} a_{\mathbf{f}, \mathbf{f}_1} L_i \text{Std}(\underline{w}^{\mathbf{f}}, \underline{y}^{\mathbf{f}_1}).$$

where the first equality follows from the construction of G , mentioned above. On the other hand, for sequences $\mathbf{f} = (f_1, f_2, \dots, f_k)$ and \mathbf{f}_1 we have

$$L_i \text{Std}(\underline{w}^{\mathbf{f}}, \underline{y}^{\mathbf{f}_1}) = \begin{cases} s_{j_1}^{f_1} s_{j_2}^{f_2} \dots s_{j_{i-1}}^{f_{i-1}} \alpha_i \text{Std}(\underline{w}^{\mathbf{f}}, \underline{y}^{\mathbf{f}_1}) & \text{if } f_i = 0 \\ 0 & \text{if } f_i = 1 \end{cases} \quad (6.8)$$

as one gets from the construction G : indeed if $f_i = 0$ then $\text{Std}(\underline{w}^{\mathbf{f}}, \underline{y}^{\mathbf{f}_1})$ will have a

bottom boundary dot  at the i 'th position and so the multiplication by L_i gives rise to the scalar α_i that is pulled to the left whereas if $f_i = 1$ then $\text{Std}(\underline{w}^{\mathbf{f}}, \underline{y}^{\mathbf{f}_1})$

will have a dashed line  at the i 'th position and so the multiplication by L_i gives zero. Hence, combining (6.6), (6.7) and (6.8) we get at least that

$$L_i \mathbb{L}\underline{w}, \mathbf{e} = \sum_{\mathbf{e}_1 \preceq \mathbf{e}} a_{\mathbf{e}_1} \mathbb{L}\underline{w}, \mathbf{e}_1 \quad (6.9)$$

for some coefficients $a_{\mathbf{e}_1} \in Q$. But on the other hand the light leaves basis is an R -basis for the cell module, and so we get that these $a_{\mathbf{e}_1}$ actually belong to R . In order to determine the coefficients $a_{\mathbf{e}}$ of this expansion, we multiply $\text{Std}(\underline{y}^{(1,1,\dots,1)}, \underline{w}^{\mathbf{e}})$ on the left of (6.9) and use (4.5) and the $*$ -version of (6.8). The Theorem follows from this. \square

Let us illustrate the Theorem on the example $\underline{w} := \text{sss}$, with $y = s$ and $\mathbf{e} = (1, 0, 0)$. The corresponding light leaves diagram

$$\text{Diagram (6.10): A light leaves diagram consisting of a blue arc connecting the top of the first strand to the top of the second strand, with a blue dot on the first strand at the bottom boundary.$$
 (6.10)

is the last one of (5.9), denoted $E4$ in (6.4) and therefore the maximal of the diagrams appearing in (5.9) with respect to path dominance. Now, applying L_1 to it gives the diagram denoted $E3$, that is

$$\text{Diagram (6.11): A light leaves diagram consisting of two parallel vertical blue strands, each with a blue dot on the first strand at the bottom boundary.$$
 (6.11)

which is in accordance with the Theorem since we have $c_{\mathbf{e}}(1) = 0$ for the content function.

Let us also calculate the action of L_2 on (6.10). Let $\delta \in \mathfrak{h}^*$ be the element coming from Demazure surjectivity. Then, using the polynomial and one-colour relations (3.4), (3.5), (3.7) we get that the action of L_2 on (6.10) is

$$\begin{aligned} & \text{Diagram (6.10)} = \text{Diagram (6.12) top row} = \text{Diagram (6.12) bottom row} \\ & \text{Diagram (6.12) top row: } \text{Diagram (6.10)} = \text{Diagram (6.12) top row} = \text{Diagram (6.12) top row} - \text{Diagram (6.12) top row} = \\ & \text{Diagram (6.12) bottom row: } s\delta - \delta \text{Diagram (6.12) bottom row} - \text{Diagram (6.12) bottom row} - \text{Diagram (6.12) bottom row} \end{aligned} \quad (6.12)$$

which is in accordance with the Theorem since $s\delta - \delta = -\alpha = s\alpha$.

Returning to the general situation of Definition 6.1, we define $\text{Tab}(\Lambda) := \bigcup_{\lambda \in \Lambda} \text{Tab}(\lambda)$. We then extend the partial orders \leq on the $\text{Tab}(\lambda)$'s to a partial order on $\text{Tab}(\Lambda)$ by the rule: $s < t$ if either $s, t \in \text{Tab}(\lambda)$ and $s < t$ or if $s \in \text{Tab}(\lambda), t \in \text{Tab}(\mu)$ and $\lambda < \mu$.

With this notation, Mathas formulated the following *separation* condition in [Ma].

Definition 6.3. Let $\mathbf{L} = \{L_1, L_2, \dots, L_k\}$ be a family of JM-elements as in Definition 6.1. Then \mathbf{L} is said to separate $\text{Tab}(\Lambda)$ if for all $s, t \in \text{Tab}(\lambda)$ satisfying $s < t$, there exist i such that $c_s(i) \neq c_t(i)$.

An important consequence of the separation condition in the following Proposition, also to be found in for example [Ma].

Proposition 6.4. Suppose that \mathbf{L} is a family of JM-elements for a cellular algebra A . If \mathbf{L} satisfies the separation condition, then A is semisimple.

Our next result is an application of this theory.

Proposition 6.5. The family of JM-elements \mathbf{L} from Theorem 9.5 satisfies the separation condition. In particular $A_{\underline{w}}$ is semisimple over Q .

Proof: Let $\mathbf{e} = (e_1, \dots, e_k) \in \text{Tab}(y)$ and $\mathbf{f} = (f_1, \dots, f_k) \in \text{Tab}(y_1)$. Let i be minimal such that $e_i \neq f_i$. Then the symbols for \mathbf{e} and \mathbf{f} at position i is either U in both cases or D in both cases. Thus, either one of them is $U0$ and the other $U1$ or one of them is $D0$ and the other $D1$. It now follows from definition (6.5) that the contents are different. \square

Remark. Even though the diagram categories are not Abelian and semisimplicity therefore is not defined in them, the Proposition can also be deduced directly from Theorem 4.2 and (4.5).

7. DETERMINANT FORMULA

For a general cellular algebra A with datum (Λ, Tab, C) there is a canonical family of A -modules $\{\Delta(\lambda) | \lambda \in \Lambda\}$, called the cell modules for A . Moreover, each cell module $\Delta(\lambda)$ is equipped with a natural symmetric, bilinear, A -invariant form $\langle \cdot, \cdot \rangle_\lambda$. Over a field, these forms can be used to classify the irreducible modules for A . Indeed the irreducible modules are in correspondence with the set

$$\Lambda_0 := \{\lambda \in \Lambda | \langle \cdot, \cdot \rangle_\lambda \neq 0\}.$$

For $\lambda \in \Lambda_0$ the corresponding irreducible module is given by $L(\lambda) := \Delta(\lambda) / \text{rad} \langle \cdot, \cdot \rangle_\lambda$ where $\text{rad} \langle \cdot, \cdot \rangle_\lambda$ is the radical in the usual sense of a bilinear form; it is an A -submodule of $\Delta(\lambda)$ because of the A -invariance of $\langle \cdot, \cdot \rangle_\lambda$.

Specializing to the case $A_{\underline{w}}$, the cell module $\Delta_{\underline{w}}(y)$ has basis $\{\mathbb{L}_{\underline{w}, \mathbf{e}} | \mathbf{e} \in \text{Tab}(y)\}$. Let $\underline{y} \in \mathbf{rexp}_s$ be a reduced expression for \underline{w}^e . Then the value of the form $\langle \mathbb{L}_{\underline{w}, \mathbf{e}}, \mathbb{L}_{\underline{w}, \mathbf{e}_1} \rangle_y$ is by definition the coefficient of $\mathbb{L}_{\underline{y}, (1, \dots, 1), (1, \dots, 1)}$ when $\mathbb{L}_{\underline{w}, \mathbf{e}} \mathbb{L}_{\underline{w}, \mathbf{e}_1}^*$ is expanded in the light leaves basis. As usual, $\mathbb{L}_{\underline{y}, (1, \dots, 1), (1, \dots, 1)}$ is not uniquely defined, but one possible choice is the identity. In any case, the radical of the form is independent of the particular choice of $\mathbb{L}_{\underline{y}, (1, \dots, 1), (1, \dots, 1)}$.

Let us illustrate the form on a couple of examples.

We consider the one color case $A_{\underline{ss}}$ where the basis of the cell module $\Delta_{\underline{ss}}(1)$ is explained in (5.3). We then get the following values of the bilinear form by taking the coefficient of the empty diagram

$$\langle \begin{array}{|c|c|} \hline \textcolor{blue}{\downarrow} & \textcolor{blue}{\downarrow} \\ \hline \end{array}, \begin{array}{|c|c|} \hline \textcolor{blue}{\downarrow} & \textcolor{blue}{\downarrow} \\ \hline \end{array} \rangle_1 = \alpha^2 \quad \langle \begin{array}{|c|} \hline \textcolor{blue}{\downarrow} \\ \hline \end{array}, \begin{array}{|c|} \hline \textcolor{blue}{\downarrow} \\ \hline \end{array} \rangle_1 = 0 \quad \langle \begin{array}{|c|} \hline \textcolor{blue}{\downarrow} \\ \hline \end{array}, \begin{array}{|c|c|} \hline \textcolor{blue}{\downarrow} & \textcolor{blue}{\downarrow} \\ \hline \end{array} \rangle_1 = \alpha. \quad (7.1)$$

Thus the determinant $\det\langle\cdot,\cdot\rangle_1$ of $\langle\cdot,\cdot\rangle_1$ is $-\alpha^2$. In particular, since $-\alpha^2$ is a unit in Q we get that $L_{\underline{ss}}(1) = \Delta_{\underline{ss}}(1)$. Of course we already knew this from the semisimplicity results of the previous section. On the other hand, if α were specialized to 0 in the ground field then $\langle\cdot,\cdot\rangle_1 = 0$ and we would have $1 \notin \Lambda_0$.

We then consider $A_{\underline{ss}}$ and the cell module $\Delta_{\underline{ss}}(s)$ whose basis is explained in (5.4). We then get the following values of the bilinear form

$$\langle \begin{array}{|c|c|} \hline \textcolor{blue}{\downarrow} & \textcolor{blue}{\downarrow} \\ \hline \end{array}, \begin{array}{|c|c|} \hline \textcolor{blue}{\downarrow} & \textcolor{blue}{\downarrow} \\ \hline \end{array} \rangle_{s_1} = \alpha \quad \langle \begin{array}{|c|} \hline \textcolor{blue}{\downarrow} \\ \hline \end{array}, \begin{array}{|c|} \hline \textcolor{blue}{\downarrow} \\ \hline \end{array} \rangle_{s_1} = 0 \quad \langle \begin{array}{|c|} \hline \textcolor{blue}{\downarrow} \\ \hline \end{array}, \begin{array}{|c|c|} \hline \textcolor{blue}{\downarrow} & \textcolor{blue}{\downarrow} \\ \hline \end{array} \rangle_{s_1} = 1. \quad (7.2)$$

Thus the determinant $\det\langle\cdot,\cdot\rangle_s$ of $\langle\cdot,\cdot\rangle_s$ is 1 and $\Delta_{\underline{ss}}(s)$ is irreducible, even in specializations.

The bilinear forms we are here considering coincide with the intersection forms from [JW] and [Wi], see also [EW] in the bimodule setting, where there are more examples to be found. In these references, the forms are further decomposed using the grading. Our diagonalization methods, to be explained next, only work in the ungraded case.

In complete generality the form $\langle\cdot,\cdot\rangle_\lambda$ is intractable, but for a cellular algebra with a separating family \mathbf{L} of JM-elements, there is at least a diagonalization strategy for calculating $\det\langle\cdot,\cdot\rangle_\lambda$.

Let us recall it, using [Ma] as reference. Let A_K be the cellular algebra in consideration, defined over the field K , and admitting a separating family \mathbf{L} of JM-elements. Let the cell modules be $\Delta_K(\lambda)$ for $\lambda \in \Lambda$. Suppose that $s \in \text{Tab}(\lambda)$ and set

$$F_s := \prod_i \prod_{c \neq c_s(i)} \frac{L_i - c}{c_s(i) - c} \in A_K. \quad (7.3)$$

Then we have $L_i F_s := c_s(i) F_s$ for all i and

$$\sum_{s \in \text{Tab}(\lambda)} F_s = 1, \quad F_s F_t = \delta_{st} F_s \quad (7.4)$$

where δ_{st} is the Kronecker delta. Define the *seminormal basis* elements as $f_{st} := F_s C_{st}^\lambda F_t \in A_K$ for $s, t \in \text{Tab}(\lambda)$. For each $\lambda \in \Lambda$ fix some $t_0 \in \text{Tab}(\lambda)$ and define $C_s^\lambda = C_{st_0}^\lambda$ and $f_s^\lambda = f_{st_0}$. Then $\{f_s^\lambda | s \in \text{Tab}(\lambda)\}$ is the seminormal basis for $\Delta_K(\lambda)$. Defining $\gamma_s \in K$ by

$$\langle f_s^\lambda, f_s^\lambda \rangle_\lambda = \langle f_s^\lambda, C_s^\lambda \rangle_\lambda = \gamma_s \quad (7.5)$$

we arrive at the following determinant formula

$$\det\langle\cdot,\cdot\rangle_\lambda = \prod_{s \in \text{Tab}(\lambda)} \gamma_s. \quad (7.6)$$

For $\underline{w} = s_{i_1} s_{i_2} \dots s_{i_k} \in \mathbf{exp}_s$ we now return to our cellular algebra $A_{\underline{w}}$ defined over R and let $A_{\underline{w}, Q} := A_{\underline{w}} \otimes_R Q$. In this particular case there is actually a very

direct diagrammatic way of constructing the eigenspace idempotents $F_{\mathbf{e}} \in A_{\underline{w}, Q}$ which can be easily explained using the previous sections and already appears implicitly in [EW1]. Indeed, each $\mathbf{e} \in \text{Tab}(\Lambda) = \{0, 1\}^k$ gives rise to the following diagram $F_{\mathbf{e}}^{diag}$ in $Kar(\mathcal{D}_Q)$

$$F_{\mathbf{e}}^{diag} := \frac{1}{\alpha_{i_1}} \frac{1}{\alpha_{i_2}} \cdots \frac{1}{\alpha_{i_j}} \cdots \frac{1}{\alpha_{i_{k-1}}} \frac{1}{\alpha_{i_k}} \quad (7.7)$$

$e_1 \quad e_2 \quad \quad \quad e_j \quad \quad \quad e_{k-1} \quad e_k$

where $e_j = 1$ corresponds to a dashed line at the j 'th position, whereas $e_j = 0$ corresponds to a dashed line at that position and the colors of the lines are adjusted to \underline{w} . For example, in the above picture $e_1 = 1, e_2 = 0$ and s_j is blue.

Using (4.8) we get that $\sum_{\mathbf{e}} F_{\mathbf{e}}^{diag} = 1$ and therefore we have

$$F_{\mathbf{e}}^{diag} = F_{\mathbf{e}}. \quad (7.8)$$

We may now use $F_{\mathbf{e}}^{diag}$ to calculate $\gamma_{\mathbf{e}} \in Q$. Recall that we have $\underline{w} = s_{j_1} s_{j_2} \cdots s_{j_k}$ and $\mathbf{e} \in \text{Tab}(y)$. Fix $i \in \{1, \dots, k\}$ and set $w^{<i} := s_{j_1}^{e_1} s_{j_2}^{e_2} \cdots s_{j_{i-1}}^{e_{i-1}}$. Let M be the symbol of \mathbf{e} at the i 'th position and define for $i \in \{1, \dots, k\}$ the element $\epsilon_{\mathbf{e}}^i$ by the formula

$$\epsilon_{\mathbf{e}}^i = \begin{cases} w^{<i} \alpha_{j_i} & \text{if } T = U0 \\ (w^{<i} \alpha_{j_i})^{-1} & \text{if } T = D0 \\ 1 & \text{if } T = U1 \\ -1 & \text{if } T = D1. \end{cases} \quad (7.9)$$

We next claim that $\epsilon_{\mathbf{e}}^i = \gamma_{\mathbf{e}}^i$. This gives rise to the promised determinant formula that transforms the diagram calculations from the definition of the form into a purely Coxeter group combinatorial calculation.

Proposition 7.1. *In the above notation we have that $\epsilon_{\mathbf{e}} = \gamma_{\mathbf{e}}$ and $\gamma_{\mathbf{e}} = \prod_i \epsilon_{\mathbf{e}}^i$ and so*

$$\det \langle \cdot, \cdot \rangle_y = \prod_{\mathbf{e} \in \text{Tab}(y), i \in \{1, \dots, k\}} \epsilon_{\mathbf{e}}^i.$$

Proof: Let us first consider the A_2 example $\mathbf{e} = (1, 0, 0)$ where $S := \{\alpha, \beta\}$, $m_{\alpha\beta} = 3$ and $\underline{w} := s_1 s_2 s_1$. The corresponding light leaves diagram is given by $(U1, U0, D0)$ and is

$$\text{Diagram} \quad (7.10)$$


In order to calculate $\gamma_{\mathbf{e}}$ we should take the coefficient of the identity in the following diagram

$$\frac{1}{\alpha} \frac{1}{\alpha} \frac{1}{\alpha} = \frac{1}{\alpha} \alpha \frac{1}{\alpha} = \alpha \frac{1}{\alpha} + \frac{1}{\alpha} \alpha \frac{1}{\alpha} \quad (7.11)$$

where we used (4.12) for the last equality. We get for $\gamma_{\mathbf{e}}$ the result $s(\alpha \frac{1}{\alpha})$, in accordance with the claim.

Let us now consider a general \mathbf{e} and let e_i be the i 'th coordinate \mathbf{e} . If the corresponding symbol is $M = U0$, then the situation is exactly as for the e_2 -term

of (7.11), and the contribution to $\gamma_{\mathbf{e}}$ is $w^{<i}\alpha_i$ as claimed. If the corresponding symbol is $M = D0$, then the situation is exactly as for the e_3 -term of (7.11), adding a trivalent vertex and the contribution to $\gamma_{\mathbf{e}}$ is $(w^{<i}\alpha_i)^{-1}$ as claimed.

On the other hand, if $T = U1$ then there are two possibilities. Either the situation is as for e_1 in (7.11) where the dashed line extends to the top and bottom of the diagram and so the contribution to $\gamma_{\mathbf{e}}$ will be 1, as claimed. Or alternatively, the dashed line turns around and eventually runs into a later $D1$. This situation is not represented in (7.11) but appears in the simple example $\underline{w} := \textcolor{blue}{s}s$, $\mathbf{e} = (1, 1)$ with symbols $(U1, D1)$. The light leaves morphism is in this case  and so we should take the coefficient of the empty diagram in

$$\begin{array}{c} \frac{1}{\alpha} \quad \frac{1}{\alpha} \\ \vdots \quad \vdots \\ \text{---} \end{array} \quad (7.12)$$

By (4.1), (4.2), (4.11) and (4.12) the value of this is -1 , which is therefore the combined contribution of the $U1$ and $D1$ symbols in this case, as claimed. The general case is done the same way. \square

Remark. Note that $\gamma_{\mathbf{e}}^i \in Q$, but even so $\det\langle \cdot, \cdot \rangle_y \in R$, of course.

Example. Let us take $\underline{w} := \textcolor{blue}{s}s$. In (5.8) we have given the basis for $\Delta(1)$ and in (5.9) the basis for $\Delta(s)$. But using the symbols given in (5.7) we get easily via the Proposition, without drawing diagrams, that $\det\langle \cdot, \cdot \rangle_1 = \alpha^4$ and $\det\langle \cdot, \cdot \rangle_s = 1$.

Remark. It follows from the Proposition that $\det\langle \cdot, \cdot \rangle_y$ is always a product of roots.

Remark. As already mentioned in the introduction our methods do not take the grading on \mathcal{D} into account.

8. A SHAPOVALOV-LIKE EXPRESSION FOR $\det\langle \cdot, \cdot \rangle_y$.

The determinant expressions are most useful in representation theory if they can be rewritten in terms of characters or dimensions of other cell modules. This was done first by Shapovalov and Jantzen for Vema modules for complex semisimple Lie algebras, see [Sh] and [Ja]. Later, similar rewritings were found for Weyl modules for algebraic groups by Jantzen and Andersen, for Specht modules for the symmetric groups by Schaper, James, Murphy and so on, see [A], [A1], [JM]. This section is devoted to a rewriting of $\det\langle \cdot, \cdot \rangle_y$ in this spirit. A major difference between our case and the above mentioned cases is that we do not have a closed formula for the dimension of the cell module.

Our argument will be an induction on the length of \underline{w} using Proposition 7.1. In this sense, it is closest to the argument given by James and Murphy, [JM], in the symmetric group case. Just like in [JM] we also need a branching rule for $\Delta_{\underline{w}}(y)$. In our setting it was discovered by D. Plaza in [P], at least in the bimodule setting.

Recall that $\underline{w} = s_{j_1} \dots s_{j_k}$ is a fixed expression. Set $\alpha := \alpha_{j_k}$ and $s := s_{j_k}$, where j_k corresponds to the color blue, say. For any $x \in W$ we define $x' := x\textcolor{blue}{s}$. There is a straightforward embedding $A_{\underline{w}'} \subseteq A_{\underline{w}}$ defined diagrammatically by adding a through blue line to the right of a diagram in $A_{\underline{w}'}$, as illustrated below

$$\begin{array}{|c|} \hline \times \\ \hline \end{array} \mapsto \begin{array}{|c|} \hline \times \\ \hline \end{array} \quad \textcolor{blue}{|} \quad (8.1)$$

Letting $A_{\underline{w}'}$ -mod and $A_{\underline{w}}$ -mod denote the module categories for $A_{\underline{w}'}$ and $A_{\underline{w}}$ we get an associated restriction functor

$$A_{\underline{w}}\text{-mod} \rightarrow A_{\underline{w}'}\text{-mod}, \quad M \mapsto \text{Res } M. \quad (8.2)$$

In general, we use from now on the convention that $\Delta_{\underline{z}}(y) := 0$ if $y \not\leq z$. The following branching rule was found by D. Plaza in the bimodule setting, see [P1]. We here state and prove it in the diagrammatical setting.

Theorem 8.1. *Assume that $y \leq w$.*

1) If $y' > y$ then there is a short exact sequence of $A_{\underline{w}'}$ -modules





$$0 \rightarrow \Delta_{\underline{w}'}(y) \rightarrow \text{Res } \Delta_{\underline{w}}(y) \rightarrow \Delta_{\underline{w}'}(y') \rightarrow 0. \quad (8.3)$$

2) If $y' < y$ then there is a short exact sequence of $A_{\underline{w}'}$ -modules

$$0 \rightarrow \Delta_{\underline{w}'}(y') \rightarrow \text{Res } \Delta_{\underline{w}}(y) \rightarrow \Delta_{\underline{w}'}(y) \rightarrow 0. \quad (8.4)$$

Remark. In [P] the sequences are considered as sequences of graded modules over the graded algebra $A_{\underline{w}'}$. In view of the last remark of the previous section we here ignore the grading.

Proof: Let first M be the symbol at the k 'th position of any $\mathbf{e} \in \text{Tab}(y)$. By definition we have that if $y' > y$, then M is either $U0$ or $D1$ whereas if $y' < y$, then M is either $U1$ or $D0$.

Let us first consider the case $y' > y$. The light leaves diagrams $\mathbb{LL}_{\underline{w}, \mathbf{e}}$ satisfying $M = U0$ are exactly those with  on the right. They define an $A_{\underline{w}'}$ -module isomorphic to $\Delta_{\underline{w}'}(y)$ where the isomorphism is given by adding  to the right. On the other hand, the light leaves diagrams $\mathbb{LL}_{\underline{w}, \mathbf{e}}$ satisfying $M = D1$ are in bijection with the light leaves basis for $\Delta_{\underline{w}'}(y')$ where the bijection can be described by bending up the last line, thus transforming a  to a through blue line . For example, in the $A1$ -case given in (5.8) the first two diagrams end in $M = U0$ and the last two diagrams end in $M = D1$ and these become, after bending up, the diagrams of (5.4).

The bending up also works well at the module level. Indeed, let us extend it to a linear map $\pi : \Delta_w(y) \rightarrow \Delta_{w'}(y')$ given diagrammatically via

$$\pi : \begin{array}{c} | \\ | \\ \text{---} \square \text{---} \\ | \\ | \end{array} \mapsto 0, \quad \begin{array}{c} | \\ | \\ \text{---} \square \text{---} \\ | \\ | \end{array} \mapsto \begin{array}{c} | \\ | \\ \text{---} \square \text{---} \\ | \\ | \end{array}. \quad (8.5)$$

Then $\ker \pi = \Delta_{\underline{w}'}(y)$, $\text{im } \pi = \Delta_{\underline{w}'}(y)$. Moreover, π is an $A_{\underline{w}'}$ -linear map as follows from the diagrammatical identity

$$\begin{array}{|c|} \hline \text{X} \\ \hline \text{X} \\ \hline \text{X} \\ \hline \end{array} = \begin{array}{|c|} \hline \text{X} \\ \hline \text{X} \\ \hline \text{X} \\ \hline \end{array} \quad (8.6)$$

where the left hand side represents $\pi(av)$ for $a \in A_{\underline{w'}}$ and $v \in \Delta_{\underline{w}}(y)$ and the right hand side $a\pi(v)$. Upon expanding the left hand side in terms of the light leaves

basis for $\Delta_{\underline{w}}(y)$ the terms with a blue dot in the upper right corner are mapped to zero under π and are also viewed as zero in the right hand side.

In the case $y' < y$, the \mathbf{e} 's with $M = U1$ correspond to diagrams with a blue line \downarrow to the right. These are in correspondence with a basis for the $A_{\underline{w}'}$ -module $\Delta_{\underline{w}'}(y')$, where the correspondence is given by deleting the blue line. The remaining diagrams, corresponding to $M = D0$, have a trivalent vertex on the right. These diagrams are in bijection with the light leaves basis for $\Delta_{\underline{w}'}(y)$, where the bijection is given by transforming the trivalent vertex to a line.

Also in this case, one checks that the maps work well at the module level. \square

In order to give the Shapovalov-like reformulation of the determinant, we shall assume that \mathfrak{h} is one of the following types. Either it is the geometric representation V of (W, S) , considered in [H1]. Recall that V is a realization of (W, S) , in the above sense, defined over \mathbb{R} and faithful as a W -representation. A root $\alpha \in V^*$ is an element of the form $\alpha = w\alpha_s$ for $s \in S$ and $w \in W$. It is said to be positive, written $\alpha > 0$, if $\alpha = \sum_{s \in S} \lambda_s \alpha_s$ for $\lambda_s \geq 0$ for all s . For α a root there is a well-defined associated reflection s_α defined by $s_\alpha = zsz^{-1}$ where $z \in W$ is chosen such that $\alpha = z\alpha_s$.

For our second choice we shall assume that (W, S) is a Weyl group. We shall once again start with the geometric representation V of W . Let $p > 2$. Since V is defined over \mathbb{Z} we obtain a realization of (W, S) by tensoring with the field $\overline{\mathbb{F}_p}$, that is $\mathfrak{h} := V \otimes \overline{\mathbb{F}_p}$. If $\beta \in V$ we also write β for its image $\beta \otimes 1$ in $V \otimes \overline{\mathbb{F}_p}$, this should not cause confusion. In V we have the concept of roots and positive roots just as before. We say that $\beta \in V \otimes \overline{\mathbb{F}_p}$ is a root if the preimage $\beta \in V$ is a root and similarly we introduce the meaning of positive roots in $V \otimes \overline{\mathbb{F}_p}$.

Although other choices for \mathfrak{h} are possible, at present the above ones are sufficient for the applications we have in mind. We remark that \mathfrak{h} is reflection vector faithful in the above cases and therefore we have equivalences to the bimodules categories, see for example the introduction of [EL].

Let us now return to the determinant $\det\langle \cdot, \cdot \rangle_\lambda$. We prove the following Theorem, giving the promised Shapovalov-like reformulation of the determinant.

Theorem 8.2. *Let \mathfrak{h} be either V or $V \otimes \overline{\mathbb{F}_p}$ as explained above. Then we have that*

$$\det\langle \cdot, \cdot \rangle_y = \pm \prod_{\beta > 0, s_\beta y > y} \beta^{\dim \Delta_{\underline{w}}(s_\beta y)} \quad (8.7)$$

where the formula for $V \otimes \overline{\mathbb{F}_p}$ is viewed as the formula for V , reduced modulo p .

Proof: It is enough to prove the case $\mathfrak{h} = V$ since the determinant in the case $\mathfrak{h} = V \otimes \overline{\mathbb{F}_p}$ is simply reduction modulo p of the determinant in the case $\mathfrak{h} = V$.

The proof is a purely combinatorial transformation of the formula from Proposition 7.1 into the expression given in (8.7), using induction on the length k of \underline{w} . The case $k = 1$ is straightforward to check. Let us therefore assume that (8.7) holds for $k - 1$ and verify it for k . Suppose first that $y' > y$. As mentioned in the proof of the previous Theorem 8.1, the \mathbf{e} 's ending in $U0$ give rise to a basis for $\Delta_{\underline{w}}(y)$, whereas those ending in $D1$ give rise to a basis for $\Delta_{\underline{w}'}(y')$, by restriction to \underline{w}' . Thus, by the inductive hypothesis and Proposition 7.1 we get that

$$\det\langle \cdot, \cdot \rangle_y = \pm (y\alpha)^{\dim \Delta_{\underline{w}'}(y)} \prod_{\beta > 0, s_\beta y > y} \beta^{\dim \Delta_{\underline{w}'}(s_\beta y)} \prod_{\beta > 0, s_\beta y' > y'} \beta^{\dim \Delta_{\underline{w}'}(s_\beta y')} \quad (8.8)$$

Now using (8.3) and (8.4) we get that

$$\dim \Delta_{\underline{w}}(s_{\beta}y) = \dim \Delta_{\underline{w}'}(s_{\beta}y') + \dim \Delta_{\underline{w}'}(s_{\beta}y) \quad (8.9)$$

and so we would like to join the two last products of (8.8) in one product, running over $\beta > 0$ with $s_{\beta}y > y$. As long as β is in $R_1 := \{\beta > 0 | s_{\beta}y > y \text{ iff } s_{\beta}y' > y'\}$ this is clearly possible. Let us therefore analyse the sets

$$R_2 := \{\beta > 0 | s_{\beta}y < y \text{ and } s_{\beta}y' > y'\} \text{ and } R_3 := \{\beta > 0 | s_{\beta}y > y \text{ and } s_{\beta}y' < y'\}.$$

We first claim that $R_2 = \emptyset$. Indeed, if $\beta \in R_2$, then $s_{\beta}y < y$ and so we get that $l(s_{\beta}y) \leq l(y) - 1 = l(y') - 2$, in contradiction with $s_{\beta}y' > y'$.

Let us then consider $\beta \in R_3$. Let $y = s_{i_1} \dots s_{i_m} s$ be a reduced expression for y . Then $y' = s_{i_1} \dots s_{i_m} s'$ is a reduced expression for y' . By the strong exchange condition, see Theorem 5.8 of [H1], applied to $s_{\beta}y' < y'$, we get that either $s_{\beta}y' = s_{i_1} \dots \widehat{s_{i_j}} \dots s_{i_m} s'$ for some j , or $s_{\beta}y' = s_{i_1} \dots s_{i_m}$. The first case is impossible since it would imply that $s_{\beta}y = s_{i_1} \dots \widehat{s_{i_j}} \dots s_{i_m} < y$. The second case corresponds to $s_{\beta}ys = y$, or equivalently $s_{\beta} = ysy^{-1}$ and so $R_3 = \{y\alpha\}$.

All in all, using (8.9) we find that the contribution to (8.8) from R_1 is

$$\pm \prod_{\beta \in R_1, s_{\beta}y > y} \beta^{\dim \Delta_{\underline{w}'}(s_{\beta}y)} \prod_{\beta \in R_1, s_{\beta}y' > y'} \beta^{\dim \Delta_{\underline{w}'}(s_{\beta}y')} = \pm \prod_{\beta \in R_1} \beta^{\dim \Delta_{\underline{w}}(s_{\beta}y)}. \quad (8.10)$$

The contribution from R_3 to (8.8) comes from the second factor. But since $s_{\beta}y = y'$, its contribution together with the first factor of (8.8) is

$$(y\alpha)^{\dim \Delta_{\underline{w}'}(y)} (y\alpha)^{\dim \Delta_{\underline{w}'}(y')} = (y\alpha)^{\dim \Delta_{\underline{w}}(y)}. \quad (8.11)$$

Finally, since R_2 does not contribute and since $R_1 \cup R_3 = \{\beta > 0 | s_{\beta}y > y\}$ we conclude that our formula (8.7) of the Theorem holds.

The case $y' < y$ is treated the same way. \square

9. JANTZEN TYPE FILTRATIONS AND SUM FORMULAS.

We construct in this section an analogue of the Jantzen filtration and, as an application of Theorem 8.2, we obtain its associated sum formula. Recall that the original Jantzen filtration was constructed in the setting of Verma modules for semisimple complex Lie algebras. Although the combinatorial methods used here have little in common with those used in the theory of Verma modules, ultimately the sum formulas rely in both cases on determinant expressions.

Let us first suppose that $\mathfrak{h} := V$ is the geometric representation of (W, S) , defined over the real numbers \mathbb{R} . Let \mathcal{R} be the polynomial algebra $\mathcal{R} := \mathbb{R}[x]$. Then \mathcal{R} is a discrete valuation domain with maximal ideal (x) and x -adic valuation $\nu_x(\cdot)$. Recall that $R = \oplus_m S^m(V^*)$. It may be identified with the polynomial algebra over \mathbb{R} in $|S|$ variables and hence there is a natural algebra homomorphism $\varphi : R \rightarrow \mathcal{R}$ satisfying $\nu_x(\varphi(\beta)) = 1$ for all roots β .

Let us consider the base change from R to \mathcal{R}

$$A_{\underline{w}, \mathcal{R}} := A_{\underline{w}} \otimes_R \mathcal{R}. \quad (9.1)$$

By general theory we then know that $A_{\underline{w}, \mathcal{R}}$ is a cellular algebra as well, with cell modules $\Delta_{\underline{w}, \mathcal{R}}(y) := \Delta_{\underline{w}}(y) \otimes_R \mathcal{R}$. Let us denote the associated bilinear form on $\Delta_{\underline{w}, \mathcal{R}}(y)$ by $\langle \cdot, \cdot \rangle_y$ as well. Motivated by Jantzen's original work we now define

$$\Delta_{\underline{w}, \mathcal{R}}^i(y) := \{v \in \Delta_{\underline{w}, \mathcal{R}}(y) | \langle v, w \rangle_y \in x^i \mathcal{R} \text{ for all } w \in \Delta_{\underline{w}, \mathcal{R}}(y)\}. \quad (9.2)$$

Since $\langle \cdot, \cdot \rangle_y$ is $A_{\underline{w}, \mathcal{R}}$ -invariant this gives a decreasing filtration

$$\Delta_{\underline{w}, \mathcal{R}}(y) = \Delta_{\underline{w}, \mathcal{R}}^0(y) \supseteq \Delta_{\underline{w}, \mathcal{R}}^1(y) \supseteq \Delta_{\underline{w}, \mathcal{R}}^2(y) \supseteq \cdots \supseteq \Delta_{\underline{w}, \mathcal{R}}^N(y) = 0 \quad (9.3)$$

of $A_{\underline{w}, \mathcal{R}}$ -modules where N is some big enough integer.

From \mathcal{R} we can further extend scalars to \mathbb{R} via the homomorphism $\mathcal{R} \rightarrow \mathcal{R}/(x) \cong \mathbb{R}$. We denote the corresponding cellular algebra by $A_{\underline{w}, \mathbb{R}}$ and its cell modules by $\Delta_{\underline{w}, \mathbb{R}}(y) := \Delta_{\underline{w}, \mathcal{R}}(y) \otimes_{\mathcal{R}} \mathbb{R}$. We denote the bilinear form on $\Delta_{\underline{w}, \mathbb{R}}(y)$ by $\langle \cdot, \cdot \rangle_y$, too.

In [P] the cellular algebra $A_{\underline{w}, \mathbb{R}}$ is studied. Using Elias and Williamson's proof of the Soergel conjecture, it was shown in [P] that the graded decomposition numbers for $A_{\underline{w}, \mathbb{R}}$ are Kazhdan-Lusztig polynomials for W or 0. In particular $A_{\underline{w}, \mathbb{R}}$ is a highly interesting non-semisimple algebra. Even if W is a Weyl group it is not quasi-hereditary, but still has many features in common with category \mathcal{O} of a complex semisimple Lie algebra.

The inclusion $\Delta_{\underline{w}, \mathcal{R}}^i(y) \subseteq \Delta_{\underline{w}, \mathcal{R}}(y)$ induces a homomorphism $\Delta_{\underline{w}, \mathbb{R}}^i(y) \rightarrow \Delta_{\underline{w}, \mathbb{R}}(y)$ and we define $\Delta_{\underline{w}, \mathbb{R}}^i(y) \subseteq \Delta_{\underline{w}, \mathbb{R}}(y)$ as the image of this homomorphism. Via (9.4) this gives rise to a filtration of $A_{\underline{w}, \mathbb{R}}$ -modules

$$\Delta_{\underline{w}, \mathbb{R}}(y) = \Delta_{\underline{w}, \mathbb{R}}^0(y) \supseteq \Delta_{\underline{w}, \mathbb{R}}^1(y) \supseteq \Delta_{\underline{w}, \mathbb{R}}^2(y) \supseteq \cdots \supseteq \Delta_{\underline{w}, \mathbb{R}}^N(y) = 0. \quad (9.4)$$

This filtration of $\Delta_{\underline{w}, \mathbb{R}}(y)$ is an analogue of the Jantzen filtration for Verma modules and the following is a weak analogue of Jantzen's sum formula, involving dimensions of the modules.

Theorem 9.1. *We have that $\dim \Delta_{\underline{w}, \mathbb{R}}(y) / \Delta_{\underline{w}, \mathbb{R}}^1(y) = \dim \Delta_{\underline{w}, \mathbb{R}}(y) / \text{rad} \langle \cdot, \cdot \rangle_y$. Moreover, the following sum formula holds*

$$\sum_{i > 0} \dim \Delta_{\underline{w}, \mathbb{R}}^i(y) = \sum_{\beta > 0, s_\beta y > y} \dim \Delta_{\underline{w}, \mathbb{R}}(s_\beta y). \quad (9.5)$$

Proof: The proof follows closely the proof in the classical case, see 'Key Lemma' of section 5.6 of [H]. For the reader's convenience we here sketch the argument. We first prove (9.5). Let $s := \text{rank } \Delta_{\underline{w}, \mathcal{R}}(y)$. Since \mathcal{R} is a principal ideal domain, the matrix M for $\langle \cdot, \cdot \rangle_y$ is equivalent to a diagonal \mathcal{R} -matrix $D := \text{diag}(d_1, \dots, d_s)$, in other words there are invertible \mathcal{R} -matrices A, B of sizes $s \times s$ such that $D = AMB$. For $j = 1, \dots, s$ we let v_j be the basis vector corresponding to the j 'th column of D and we set $a_j := \nu_x(d_j)$. Then $\Delta_{\underline{w}, \mathcal{R}}^i(y)$ is spanned by the elements $\{v_j | a_j \geq i\}$ together with the elements $\{x^{i-a_j} v_j | a_j < i\}$. The last elements vanish when tensoring over \mathbb{R} and so the left hand side of (9.5) is $a_1 + a_2 + \dots + a_s$. But this is equal to $\nu_x(\det D)$ which is equal to the right hand side of (9.5) by Theorem 8.7 and the assumption on φ .

The equality $\dim \Delta_{\underline{w}, \mathbb{R}}(y) / \Delta_{\underline{w}, \mathbb{R}}^1(y) = \dim \Delta_{\underline{w}, \mathbb{R}}(y) / \text{rad} \langle \cdot, \cdot \rangle_y$ is proved similarly. \square

We next explain one of several ways to obtain a sum formula in the positive characteristic situation. Let $\mathfrak{h} := V$ be the geometric representation of the Weyl group W , as above. Let p be a prime number and choose this time $\mathcal{R} := \mathbb{Z}_p$, the localization of \mathbb{Z} at p . This \mathcal{R} is also a discrete valuation domain with valuation function denoted $\nu_p(\cdot)$. Its maximal ideal is $p\mathcal{R}$ and we have $\mathcal{R}/p\mathcal{R} \cong \mathbb{F}_p$. Let $\varphi : V \rightarrow \mathbb{Z}$ be the homomorphism of \mathbb{Z} -modules given by $\varphi(\alpha_s) := p$ for the basis elements $\{\alpha_s, s \in S\}$ and denote also by φ the homomorphism obtained from φ by composing with the inclusion $\mathbb{Z} \subseteq \mathcal{R}$.

We define $A_{\underline{w}, \mathcal{R}} := A_{\underline{w}} \otimes_{\mathcal{R}} \mathcal{R}$, $A_{\underline{w}, \mathbb{F}_p} := A_{\underline{w}, \mathcal{R}} \otimes_{\mathcal{R}} \mathbb{F}_p$ and so on, mimicking what we did before. We then obtain a $A_{\underline{w}, \mathcal{R}}$ -module filtration of $\Delta_{\underline{w}, \mathcal{R}}(y)$ via

$$\Delta_{\underline{w}, \mathcal{R}}^i(y) := \{v \in \Delta_{\underline{w}, \mathcal{R}}(y) \mid \langle v, w \rangle_y \in p^i \mathcal{R} \text{ for all } w \in \Delta_{\underline{w}, \mathcal{R}}(y)\} \quad (9.6)$$

and this induces a filtration $\{\Delta_{\underline{w}, \mathbb{F}_p}^i(y) \mid i = 0, 1, \dots\}$ of $\Delta_{\underline{w}, \mathbb{F}_p}(y)$ and hence also a filtration $\{\Delta_{\underline{w}, \mathbb{F}_p}^i(y) \mid i = 0, 1, \dots\}$ of $\Delta_{\underline{w}, \mathbb{F}_p}(y)$ via $\Delta_{\underline{w}, \mathbb{F}_p}^i(y) := \Delta_{\underline{w}, \mathbb{F}_p}^i(y) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}_p}$. We get the following Theorem.

Theorem 9.2. *We have that $\dim \Delta_{\underline{w}, \mathbb{F}_p}(y) / \Delta_{\underline{w}, \mathbb{F}_p}^1(y) = \dim \Delta_{\underline{w}, \mathbb{F}_p}(y) / \text{rad} \langle \cdot, \cdot \rangle_y$. Moreover, the following sum formula holds*

$$\sum_{i \geq 0} \dim \Delta_{\underline{w}, \mathbb{F}_p}^i(y) = \sum_{\beta > 0, s_\beta y > y} \nu_p(\varphi(\beta)) \dim \Delta_{\underline{w}, \mathbb{F}_p}(s_\beta y). \quad (9.7)$$

Proof: The proof is essentially the same as the proof of Theorem 9.2. □

In the classical theory the sum formula is formulated as an equality in the Grothendieck group and thus we would have expected (9.5) and (9.7) to hold at this level of generality. Our next goal is to show that in fact there is a variation of the above constructions for which (9.5) and (9.7) do hold at the Grothendieck group level.

As can be seen already in the one-color calculations presented above, the algebra $A_{\underline{w}, \mathbb{R}}$ is not quasi-hereditary in general, in other words we have $\Lambda \neq \Lambda_0$ in general. Moreover, in general it appears to be difficult to determine Λ_0 , and hence for the Grothendieck group to work well we need to change the setup.

As mentioned above, \mathcal{D} is cellular category in the sense of Westbury. Let us recall his definition from [Wes]:

Definition 9.3. *Let \mathbb{k} be a commutative ring with identity and let \mathcal{C} be a \mathbb{k} -linear category with duality $*$. Then \mathcal{C} is called a cellular category if there exists a poset Λ and for $\lambda \in \Lambda$ and $n \in \text{Obj}(\mathcal{C})$ a finite set $\text{Tab}(n, \lambda)$ together with a map $\text{Tab}(m, \lambda) \times \text{Tab}(n, \lambda) \rightarrow \text{Hom}_{\mathcal{C}}(m, n)$, $(S, T) \mapsto C_{ST}^\lambda$, satisfying $(C_{ST}^\lambda)^* = C_{TS}^\lambda$. These data satisfy that*

$$\{C_{ST}^\lambda \mid S \in \text{Tab}(m, \lambda), T \in \text{Tab}(n, \lambda), \lambda \in \Lambda\} \text{ is a basis for } \text{Hom}_{\mathcal{C}}(m, n) \quad (9.8)$$

and for all $a \in \text{Hom}_{\mathcal{C}}(n, p)$, $S \in \text{Tab}(m, \lambda)$, $T \in \text{Tab}(n, \lambda)$

$$aC_{ST}^\lambda = \sum_{S' \in \text{Tab}(p, \lambda)} r_a(S', S) C_{S', T}^\lambda \text{ mod } A^\lambda \quad (9.9)$$

where A^λ is the span of $\{C_{ST}^\mu \mid \mu < \lambda, S \in \text{Tab}(m, \mu), T \in \text{Tab}(p, \mu)\}$.

As already mentioned in [Wes], each object m in a cellular category \mathcal{C} gives rise to the cellular algebra $\text{End}_{\mathcal{C}}(m)$. This construction can be generalized as follows. Let I be any finite subset of the objects of \mathcal{C} and define $\text{End}_{\mathcal{C}}(I)$ as the \mathbb{k} -direct sum

$$\text{End}_{\mathcal{C}}(I) := \oplus_{m, n \in I} \text{Hom}_{\mathcal{C}}(m, n). \quad (9.10)$$

Then $\text{End}_{\mathcal{C}}(I)$ has a natural \mathbb{k} -algebra structure as follows

$$g \cdot f := \begin{cases} gf & \text{if } f \in \text{Hom}_{\mathcal{C}}(m, n), g \in \text{Hom}_{\mathcal{C}}(n, p) \text{ for some } m, n, p \\ 0 & \text{otherwise.} \end{cases} \quad (9.11)$$

Theorem 9.4. *Let \mathcal{C} be a cellular category as in Definition 9.3 and define for $\lambda \in \Lambda$ the set $\text{Tab}(\lambda) := \cup_{n \in I} \text{Tab}(n, \lambda)$. Let for $S \in \text{Tab}(\lambda), T \in \text{Tab}(\lambda)$ the element $C_{ST}^\lambda \in \text{End}_{\mathcal{C}}(I)$ be defined as the corresponding inclusion of $C_{ST}^\lambda \in \text{Hom}_{\mathcal{C}}(m, n)$ in $\text{End}_{\mathcal{C}}(I)$. Then these data define a cellular algebra structure on $\text{End}_{\mathcal{C}}(I)$.*

Proof: This is immediate from the definitions. \square

Returning to our category \mathcal{D} we choose an ideal π in W , that is $x \in \pi, y < x \Rightarrow y \in \pi$, and choose for all $y \in \pi$ any $\underline{y} \in \mathbf{rep}_s$ expressing y . We set $\underline{\pi} := \{\underline{y} | y \in \pi\}$ and define the algebra

$$A_\pi = A_{\underline{\pi}} := \text{End}_{\mathcal{D}}(\oplus_{y \in \pi} \underline{y}). \quad (9.12)$$

Then, according to the above Theorem we have that A_π is a cellular algebra on the poset $\Lambda_\pi := \pi$. For $y \in \Lambda_\pi$ the corresponding $\text{Tab}(y)$ is $\text{Tab}_\pi(y) := \bigcup_{z \in \pi} \text{Tab}_z(y)$ where $\text{Tab}_z(y)$ is $\text{Tab}(y)$ for A_z .

For $y \in \pi$ we denote by $\Delta_\pi(y)$ the corresponding cell module for A_π . Its basis is given by $\bigcup_{z \in \pi} \{\mathbb{L}_{z, \mathbf{e}} | \mathbf{e} \in \text{Tab}_z(y)\}$, thus there is an R -module decomposition

$$\Delta_\pi(y) = \oplus_{z \in \pi} \Delta_z(y) \quad (9.13)$$

and in particular $\dim \Delta_\pi(y) = \sum_{z \in \pi} \dim \Delta_z(y)$. Note that a similar formula for $\dim A_\pi$ does not hold. We view (9.13) as a kind of weight space decomposition for $\Delta_\pi(y)$ with weight spaces $\Delta_z(y)$.

We point out that $\langle \cdot, \cdot \rangle_{\pi, y}$ is orthogonal with respect to the decomposition in (9.13). This is a key observation for the following.

Recall that the ground ring for A_π is R . For a field k that is made into an R -algebra via a homomorphism $R \rightarrow k$, we obtain as usual a specialized algebra $A_{\pi, k} := A_\pi \otimes_R k$. This is also a cellular algebra. We now state the result that makes us prefer A_π over $A_{\underline{w}}$.

Theorem 9.5. *A_π and $A_{\pi, k}$ are quasi-hereditary algebras.*

Proof: For $y \in \Lambda_\pi$ we must show that $\langle \cdot, \cdot \rangle_{\pi, y} \neq 0$. But since \underline{y} is a reduced expression for y we get for $\mathbf{e} = (1, 1, \dots, 1)$ that $\mathbb{L}_{\underline{y}, \mathbf{e}}$ can be chosen as the identity morphism in $\text{End}_{\mathcal{D}}(\underline{y})$ and so $\langle \mathbb{L}_{\underline{y}, \mathbf{e}}, \mathbb{L}_{\underline{y}, \mathbf{e}} \rangle_{\pi, y} = \langle \mathbb{L}_{\underline{y}, \mathbf{e}}, \mathbb{L}_{\underline{y}, \mathbf{e}} \rangle_y = 1$. The Theorem now follows. \square

Let $\mathcal{R} := \mathbb{R}[x]$ or $\mathcal{R} := \mathbb{Z}_p$ depending on $\mathfrak{h} = V$ or $\mathfrak{h} = X$ and define $A_{\pi, \mathcal{R}}, A_{\pi, \mathbb{R}}, A_{\pi, \overline{\mathbb{F}_p}}$ correspondingly. These are all quasi-hereditary algebras with corresponding cell modules $\Delta_{\pi, \mathcal{R}}(y), \Delta_{\pi, \mathbb{R}}(y)$ and $\Delta_{\pi, \overline{\mathbb{F}_p}}(y)$. We use the same notation $\langle \cdot, \cdot \rangle_{\pi, y}$ for the bilinear form on each of these modules and introduce the Jantzen type filtrations $\Delta_{\pi, \mathcal{R}}^i(y), \Delta_{\pi, \mathbb{R}}^i(y)$ and $\Delta_{\pi, \overline{\mathbb{F}_p}}^i(y)$, just as before.

We now come to the Grothendieck groups. Let either $k = \mathbb{R}$ or $k = \overline{\mathbb{F}_p}$. Let $\langle A_{\pi, k}\text{-mod} \rangle$ (resp. $\langle A_{\underline{w}, k}\text{-mod} \rangle$) be the Grothendieck groups of finite dimensional $A_{\pi, k}$ -modules (resp. finite dimensional $A_{\underline{w}, k}$ -modules). Because of quasi-heredity, the classes $[\Delta_{\pi, k}(y)]$ of the cell modules form a basis of their Grothendieck group. On the other hand, the classes $[\Delta_{\underline{w}, k}(y)]$ only form a generating set for $\langle A_{\underline{w}, k}\text{-mod} \rangle$ since $A_{\underline{w}, k}$ is not quasi-hereditary in general. But we have a projection map $\varphi_{\underline{w}}$

$$\varphi_{\underline{w}} : \langle A_{\pi, k}\text{-mod} \rangle \rightarrow \langle A_{\underline{w}, k}\text{-mod} \rangle, [\Delta_{\pi, k}(y)] \mapsto [\Delta_{\underline{w}, k}(y)]. \quad (9.14)$$

We need to describe $\varphi_{\underline{w}}$ more precisely. There is a diagonal subalgebra $\oplus_{w \in \pi} A_{\underline{w}, k}$ inside $A_{\pi, k}$ and each $A_{\underline{w}, k}$ is an algebra summand of $\oplus_{w \in \pi} A_{\underline{w}, k}$. Let

$$\tau_{\underline{w}} : A_{\pi, k}\text{-mod} \rightarrow A_{\underline{w}, k}\text{-mod}$$

be the composition of the corresponding restriction and idempotent truncation functors. Then $\tau_{\underline{w}}$ induces a \mathbb{Z} -module homomorphism between the Grothendieck groups: this is our $\varphi_{\underline{w}}$. Via this description of $\varphi_{\underline{w}}$ and the orthogonality of the decomposition (9.13) we get the following compatibility of the Jantzen filtrations

$$\varphi_{\underline{w}}([\Delta_{\pi,k}^i(y)]) = [\Delta_{\underline{w},k}^i(y)]. \quad (9.15)$$

Let now $\dim_{\underline{w}} : \langle A_{\underline{w},k}\text{-mod} \rangle \rightarrow \mathbb{Z}, [M] \mapsto \dim_k M$ be the dimension homomorphism and define a \mathbb{Z} -module homomorphism $\Phi : \langle A_{\pi,k}\text{-mod} \rangle \rightarrow \oplus_{\underline{w} \in \pi} \mathbb{Z}$ by setting the \underline{w} 'th coordinate equal to the composite $\dim_{\underline{w}} \circ \varphi_{\underline{w}}$.

Lemma 9.6. *In the above notation $\Phi : \langle A_{\pi,k}\text{-mod} \rangle \rightarrow \oplus_{\underline{w} \in \pi} \mathbb{Z}$ is an isomorphism of \mathbb{Z} -modules.*

Proof: Since $\langle A_{\pi,k}\text{-mod} \rangle$ and $\oplus_{\underline{w} \in \pi} \mathbb{Z}$ are free \mathbb{Z} -modules of the same rank, it is enough to show that Φ is surjective. Let $(n_{\underline{z}})_{\underline{z} \in \pi}$ be an element of $\oplus_{\underline{z} \in \pi} \mathbb{Z}$. Choose \underline{z}_0 satisfying $n_{\underline{z}_0} \neq 0$ and z_0 minimal in W with respect to this. The \underline{z}_0 'th component of $\Phi([\Delta_{\pi}(z_0)])$ is $\dim_k \Delta_{\underline{z}_0}(z_0) = 1$ and so the \underline{z}_0 'th component of $\Phi(n_{\underline{z}_0}[\Delta_{\pi}(z_0)])$ is $n_{\underline{z}_0}$. Moreover, the \underline{z} 'th component of $\Phi([\Delta_{\pi}(z_0)])$ is nonzero only if $z_0 \leq z$. Hence, we can use induction on $(n_{\underline{z}})_{\underline{z} \in \pi} - \Phi(n_{\underline{z}_0}[\Delta_{\pi}(z_0)])$ and get that $(n_{\underline{z}})_{\underline{z} \in \pi} \in \text{im } \Phi$ as claimed. \square

We are now in position to prove the main Theorems of this section.

Theorem 9.7. *Let $\mathfrak{h} := V$ be the geometric representation over \mathbb{R} . For the $A_{\pi,\mathbb{R}}$ -filtration $\{\Delta_{\pi,\mathbb{R}}^i(y)\}$ of $\Delta_{\pi,\mathbb{R}}(y)$ we have that $\Delta_{\pi,\mathbb{R}}(y)/\Delta_{\pi,\mathbb{R}}^1(y)$ is nonzero and irreducible and the following sum formula holds in $\langle A_{\pi,\mathbb{R}}\text{-mod} \rangle$*

$$\sum_{i>0} [\Delta_{\pi,\mathbb{R}}^i(y)] = \sum_{\beta>0, s_{\beta}y>y} [\Delta_{\pi,\mathbb{R}}(s_{\beta}y)]. \quad (9.16)$$

Proof: By the construction of the filtration, the first statement follows from the quasi-heredity of A_{π} .

To show the second statement we get by applying Φ to the left hand side of (9.16) the element of $\oplus_{\underline{w} \in \pi} \mathbb{Z}$ whose \underline{w} 'th component is $\sum_{i>0} \dim \Delta_{\underline{w},\mathbb{R}}^i(y)$. Applying Φ to the right hand side of (9.16) we get the element whose \underline{w} 'th component is $\sum_{\beta>0, s_{\beta}y>y} \dim \Delta_{\underline{w},\mathbb{R}}(s_{\beta}y)$ and so by Theorem 9.2 the two sides are equal and the Theorem follows. \square

Similarly, we have the following Theorem.

Theorem 9.8. *Let W be a Weyl group and let $\mathfrak{h} := V \otimes \overline{\mathbb{F}_p}$. For the $A_{\pi,\overline{\mathbb{F}_p}}$ -filtration $\{\Delta_{\pi,\overline{\mathbb{F}_p}}^i(y)\}$ of $\Delta_{\pi,\overline{\mathbb{F}_p}}(y)$ we have that $\Delta_{\pi,\overline{\mathbb{F}_p}}(y)/\Delta_{\pi,\overline{\mathbb{F}_p}}^1(y)$ is nonzero and irreducible and the following sum formula holds in $\langle A_{\pi,\overline{\mathbb{F}_p}}\text{-mod} \rangle$*

$$\sum_{i>0} [\Delta_{\pi,\overline{\mathbb{F}_p}}^i(y)] = \sum_{\beta>0, s_{\beta}y>y} \nu_p(\varphi(\beta)) [\Delta_{\pi,\overline{\mathbb{F}_p}}(s_{\beta}y)]. \quad (9.17)$$

An important aspect of the diagrammatical category \mathcal{D} is its \mathbb{Z} -grading. For general reasons, it induces a cellular algebra structure on its zeroth graded subcategory and then also on the zeroth graded component $A_{\underline{w}}^0$ of $A_{\underline{w}}$. Unfortunately, our L_i 's are of degree 2 and therefore do not induce JM-operators on $A_{\underline{w}}^0$ and hence, in particular, our methods do not give rise to a sum formula for $A_{\underline{w}}^0$.

10. APPLICATIONS OF THE SUM FORMULA.

We indicate in this section, via an example, how to apply formula (9.16) to obtain decomposition numbers for A_π . This is parallel to Jantzen's original calculations for Verma modules and just as in the original setting, the sum formula only gives complete information on the decomposition numbers in certain small cases.

The comparison with Verma modules is strengthened by the following analogue for A_π of the fact that homomorphisms between Verma modules are injective, even in positive characteristic. The proof of this fact relies however on the PBW-Theorem and the construction of Verma modules as induced modules from a Borel subalgebra and so it does not carry over to the present A_π -setting. To prove the statement in the A_π -setting one relies on results of D. Plaza, [P].

Theorem 10.1. *i). Let W be arbitrary and π an ideal of W . Suppose that $u, v \in W$ satisfy $u \leq v$. Then there is an embedding of $A_{\pi, \mathbb{R}}$ -modules $\Delta_{\pi, \mathbb{R}}(v) \subseteq \Delta_{\pi, \mathbb{R}}(u)$.
ii). Let W be a Weyl group and π an ideal of W . Suppose that $u, v \in W$ satisfy $u \leq v$. Let $p > 2$. Then there is an embedding of A_{π, \mathbb{F}_p} -modules $\Delta_{\pi, \mathbb{F}_p}(v) \subseteq \Delta_{\pi, \mathbb{F}_p}(u)$.*

Proof: As already mentioned above, in the settings of i) and ii) the diagrammatic categories are equivalent to the bimodule categories. In [P] it was shown in the bimodule category that $\Delta_{\underline{w}, \mathbb{R}}(v) \subseteq \Delta_{\underline{w}, \mathbb{R}}(u)$ where the embedding is given by multiplication with Deodhar's distinguished subexpression for u in v . This gives us the required embedding $\Delta_{\pi, \mathbb{R}}(v) \subseteq \Delta_{\pi, \mathbb{R}}(u)$ by summing over all relevant summands, showing i). But the argument of [P] does not depend on the characteristic of the field and so we get ii) the same way. \square

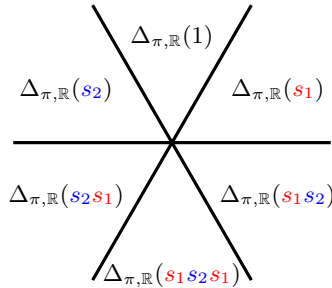
Remark. We believe that the Theorem can be proved diagrammatically, making it true for general realizations in arbitrary characteristic.

Let us illustrate our results on an example. We first take the ground field \mathbb{R} and work out the case $W = \langle s_1, s_2 \rangle$ of type A_2 , that is

$$W = \{1, s_1, s_2, s_1 s_2, s_2 s_1, s_1 s_2 s_1 = s_2 s_1 s_2\}.$$

Let us use $\pi := W$ and $\underline{\pi} := \{1, s_1, s_2, s_1 s_2, s_2 s_1, s_1 s_2 s_1\}$.

We associate with this the following alcove geometry



The images of the simple modules $[L_{\pi, \mathbb{R}}(y)]$, with y running over W , give the canonical basis of $\langle A_{\pi, \mathbb{R}}\text{-mod} \rangle$ and by general cellular algebra theory we have that

$$[\Delta_{\pi, \mathbb{R}}(y)] = \sum_{y \leq u} d_{yu}^{\mathbb{R}} [L_{\pi, \mathbb{R}}(u)] \quad (10.1)$$

where the $d_{yu}^{\mathbb{R}}$'s are the decomposition numbers. Let us determine the expansion of all $[\Delta_{\pi, \mathbb{R}}(y)]$'s in terms of $[L_{\pi, \mathbb{R}}(y)]$'s, or equivalently the decomposition numbers $d_{yu}^{\mathbb{R}}$. As a starting point we get from (10.1) that

$$\Delta_{\pi, \mathbb{R}}(\mathbf{s}_1 \mathbf{s}_2 \mathbf{s}_1) = L_{\pi, \mathbb{R}}(\mathbf{s}_1 \mathbf{s}_2 \mathbf{s}_1) \quad (10.2)$$

which gives us all the decomposition numbers $d_{yu}^{\mathbb{R}}$ with $y = \mathbf{s}_1 \mathbf{s}_2 \mathbf{s}_1$. Let us now calculate $[\Delta_{\pi, \mathbb{R}}(\mathbf{s}_2 \mathbf{s}_1)]$. By (10.1) we have that $[\Delta_{\pi, \mathbb{R}}(\mathbf{s}_2 \mathbf{s}_1)] = [L_{\pi, \mathbb{R}}(\mathbf{s}_2 \mathbf{s}_1)] + d[L_{\pi, \mathbb{R}}(\mathbf{s}_1 \mathbf{s}_2 \mathbf{s}_1)]$ for some d . The sum formula (9.16) reads in this case

$$\sum_{i>0} [\Delta_{\pi, \mathbb{R}}^i(\mathbf{s}_2 \mathbf{s}_1)] = [\Delta_{\pi, \mathbb{R}}(\mathbf{s}_1 \mathbf{s}_2 \mathbf{s}_1)] = [L_{\pi, \mathbb{R}}(\mathbf{s}_1 \mathbf{s}_2 \mathbf{s}_1)]. \quad (10.3)$$

Since $\Delta_{\pi, \mathbb{R}}^1(\mathbf{s}_2 \mathbf{s}_1) = \text{rad}\langle \cdot, \cdot \rangle_{\pi, \mathbf{s}_2 \mathbf{s}_1}$ this gives us as the only possibility $d = 1$, that is

$$[\Delta_{\pi, \mathbb{R}}(\mathbf{s}_2 \mathbf{s}_1)] = [L_{\pi, \mathbb{R}}(\mathbf{s}_2 \mathbf{s}_1)] + [L_{\pi, \mathbb{R}}(\mathbf{s}_1 \mathbf{s}_2 \mathbf{s}_1)]. \quad (10.4)$$

Similarly we have

$$[\Delta_{\pi, \mathbb{R}}(\mathbf{s}_1 \mathbf{s}_2)] = [L_{\pi, \mathbb{R}}(\mathbf{s}_1 \mathbf{s}_2)] + [L_{\pi, \mathbb{R}}(\mathbf{s}_1 \mathbf{s}_2 \mathbf{s}_1)]. \quad (10.5)$$

Let us now turn to $[\Delta_{\pi, \mathbb{R}}(\mathbf{s}_2)]$ which by (10.1) can be written as

$$[\Delta_{\pi, \mathbb{R}}(\mathbf{s}_2)] = [L_{\pi, \mathbb{R}}(\mathbf{s}_2)] + a[L_{\pi, \mathbb{R}}(\mathbf{s}_1 \mathbf{s}_2)] + b[L_{\pi, \mathbb{R}}(\mathbf{s}_2 \mathbf{s}_1)] + c[L_{\pi, \mathbb{R}}(\mathbf{s}_1 \mathbf{s}_2 \mathbf{s}_1)] \quad (10.6)$$

for some nonnegative integers a, b, c . On the other hand, the sum formula (9.16) gives in this case

$$\sum_{i>0} [\Delta_{\pi, \mathbb{R}}^i(\mathbf{s}_2)] = [\Delta_{\pi, \mathbb{R}}(\mathbf{s}_1 \mathbf{s}_2)] + [\Delta_{\pi, \mathbb{R}}(\mathbf{s}_2 \mathbf{s}_1)] = [L_{\pi, \mathbb{R}}(\mathbf{s}_1 \mathbf{s}_2)] + [L_{\pi, \mathbb{R}}(\mathbf{s}_2 \mathbf{s}_1)] + 2[L_{\pi, \mathbb{R}}(\mathbf{s}_1 \mathbf{s}_2 \mathbf{s}_1)] \quad (10.7)$$

where we for the second equality used (10.4) and (10.5). This information gives us that $a = b = 1$ but it does not determine the value of c since the filtration $\{\Delta_{\pi, \mathbb{R}}^i(\mathbf{s}_2)\}$ may have one or two nonzero terms. We next claim that

$$\dim \text{Hom}_{A_{\pi, \mathbb{R}}}(L_{\pi, \mathbb{R}}(\mathbf{s}_1 \mathbf{s}_2 \mathbf{s}_1), \Delta_{\pi, \mathbb{R}}(\mathbf{s}_2)) = 1 \quad (10.8)$$

from which it follows that there are two nonzero terms in the filtration, giving us $c = 1$. To show the claim (10.8) we define

$$A_{\text{diag}, \mathbb{R}} := A_{\mathbf{s}_1 \mathbf{s}_2 \mathbf{s}_1, \mathbb{R}} \oplus A_{\mathbf{s}_2 \mathbf{s}_1, \mathbb{R}} \oplus A_{\mathbf{s}_1 \mathbf{s}_2, \mathbb{R}} \oplus A_{\mathbf{s}_1, \mathbb{R}} \oplus A_{\mathbf{s}_2, \mathbb{R}} \oplus A_{1, \mathbb{R}}. \quad (10.9)$$

which we view as a subalgebra of $A_{\pi, \mathbb{R}}$ via the diagonal action. As an $A_{\text{diag}, \mathbb{R}}$ -module we have that

$$L_{\pi, \mathbb{R}}(\mathbf{s}_1 \mathbf{s}_2 \mathbf{s}_1) = \Delta_{\pi, \mathbb{R}}(\mathbf{s}_1 \mathbf{s}_2 \mathbf{s}_1) = \Delta_{\mathbf{s}_1 \mathbf{s}_2 \mathbf{s}_1, \mathbb{R}}(\mathbf{s}_1 \mathbf{s}_2 \mathbf{s}_1) \oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus 0 \quad (10.10)$$

and so we get by restriction that

$$\text{Hom}_{A_{\pi, \mathbb{R}}}(L_{\pi, \mathbb{R}}(\mathbf{s}_1 \mathbf{s}_2 \mathbf{s}_1), \Delta_{\pi, \mathbb{R}}(\mathbf{s}_2)) \subseteq \text{Hom}_{A_{\text{diag}, \mathbb{R}}}(L_{\pi, \mathbb{R}}(\mathbf{s}_1 \mathbf{s}_2 \mathbf{s}_1), \Delta_{\pi, \mathbb{R}}(\mathbf{s}_2)) \quad (10.11a)$$

$$\subseteq \text{Hom}_{A_{\text{diag}, \mathbb{R}}}(\Delta_{\mathbf{s}_1 \mathbf{s}_2 \mathbf{s}_1, \mathbb{R}}(\mathbf{s}_1 \mathbf{s}_2 \mathbf{s}_1) \oplus 0 \oplus \cdots \oplus 0, \Delta_{\mathbf{s}_1 \mathbf{s}_2 \mathbf{s}_1, \mathbb{R}}(\mathbf{s}_2) \oplus \cdots) \quad (10.11b)$$

which is of dimension 1 since $\dim \Delta_{\mathbf{s}_1 \mathbf{s}_2 \mathbf{s}_1, \mathbb{R}}(\mathbf{s}_1 \mathbf{s}_2 \mathbf{s}_1) = \dim \Delta_{\mathbf{s}_1 \mathbf{s}_2 \mathbf{s}_1, \mathbb{R}}(\mathbf{s}_2) = 1$.

Almost similarly we get that

$$[\Delta_{\pi, \mathbb{R}}(\mathbf{s}_1)] = [L_{\pi, \mathbb{R}}(\mathbf{s}_1)] + [L_{\pi, \mathbb{R}}(\mathbf{s}_1 \mathbf{s}_2)] + [L_{\pi, \mathbb{R}}(\mathbf{s}_2 \mathbf{s}_1)] + [L_{\pi, \mathbb{R}}(\mathbf{s}_1 \mathbf{s}_2 \mathbf{s}_1)] \quad (10.12)$$

and finally also

$$[\Delta_{\pi, \mathbb{R}}(1)] = [L_{\pi, \mathbb{R}}(1)] + [L_{\pi, \mathbb{R}}(\textcolor{red}{s}_1)] + [L_{\pi, \mathbb{R}}(\textcolor{blue}{s}_2)] + [L_{\pi, \mathbb{R}}(\textcolor{red}{s}_1 \textcolor{blue}{s}_2)] + [L_{\pi, \mathbb{R}}(\textcolor{blue}{s}_2 \textcolor{red}{s}_1)] + [L_{\pi, \mathbb{R}}(\textcolor{red}{s}_1 \textcolor{blue}{s}_2 \textcolor{red}{s}_1)]. \quad (10.13)$$

The above calculations can also be carried out in the characteristic $p > 2$ case where one gets the same decomposition numbers.

In the language of p -canonical bases, [JW], we get that for A_2 the p -canonical basis coincides with the Kazhdan-Lusztig basis. This is already proved in [JW] using very different methods.

In view of the results and conjectures of [RW], we finally remark that it would be very interesting to generalize our results to the antispherical module of affine Weyl groups.

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